ANALYSIS OF KEY COMPARISON DATA USING DETAILED UNCERTAINTY INFORMATION

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Two goals of MRA

- to provide an objective foundation for the mutual recognition of measurement and calibration certificates issued by National Metrology Institutes (NMI) according to the MRA,

to established the degree of equivalence of national measurement standards maintained by NMIs.

improve the lab's knowledge about their biases

to provide an objective foundation for the mutual recognition

- To agree on the criteria for checking the data consistency or for confirmation of the quoted uncertainties (CMC) ,
  - χ^2 criteria is usually used in key comparisons

$$\chi^2 = \sum_{1}^{n} \frac{\left(x_i - x_w\right)^2}{u^2\left(x_i\right)}$$

  - En criteria is usually used in supplementary comparisons

$$E_n = \frac{\left|x_i - x_w\right|}{2\sqrt{u^2\left(x_i\right) - u^2\left(x_w\right)}}$$

  - Others

- To organize the key comparisons or supplementary comparisons and analyze the data received

Results of data consistency checking

- If the data are consistent / equivalent the quantitative measure – DoEs - can be established

MRA approach based on KCRV concept

Other approaches.

Estimation of the labs biases are considered below

- If the data are inconsistent

checking the experiments and uncertainty budgets and repeat the analysis

the largest consistent subset (LCS) can be determined

other models for data analysis and DoEs interpretation can be proposed

MRA approach to DoEs estimation.
KCRV is a best estimate of the measurand.

Bayesian inference

- The model $$X_i \equiv X$$
- The prior $$p(x) \propto 1$$
- Data $$\{x_i, u_i\}_1^n$$
- The posterior $$p(x \mid x_1,\ldots,x_n,u_1,\ldots,u_n) \propto \prod_1^n \frac{1}{\sqrt{2\pi}\, u_i} \exp\left\{-\frac{(x_i - x)^2}{2u_i^2}\right\} p(x)$$

$$E(X) = \bar{x}_w = \left(\sum_1^n \frac{1}{u^2(x_i)}\right)^{-1} \sum_1^n \frac{x_i}{u^2(x_i)}$$

$$\mathrm{var}(X) = u^2(\bar{x}_w) = \left(\sum_1^n \frac{1}{u^2(x_i)}\right)^{-1}$$

MRA approach to DoEs estimation.

- x_i - the best estimate of the measurand obtained in th i-th lab
- x_{ref} - the best estimate of the measurand based on the information received from all labs

$$d_i = x_i - x_{ref}$$

- Difference of two estimates of the same measurand which should not exceed the associated expanded uncertainty

$$\left| x_i - x_{ref} \right| \leq U(d_i) = k\sqrt{u_i^2(x_i) - u_i^2(x_{ref})}$$

Other interpretations of the DoEs.
Bayesian inference for the lab's bias (general case) .

- The model $X_i = X + B_i$
- Bayes theorem

$$p(y, b_1, \ldots, b_N \mid x_1, \ldots, x_N) \propto l(y, b_1, \ldots, b_N \mid x_1, \ldots, x_N)\, p(y, b_1, \ldots, b_N)$$

- Joint prior distribution of the measurand and the biases

$$p(y, b_1, \ldots, b_N) \propto p(b_1) \cdots p(b_N)$$

- The likelihood

$$l(y, b_1, \ldots, b_N \mid x_1, \ldots, x_N) = l(y, b_1 \mid x_1) \cdots l(y, b_N \mid x_N)$$

- The posterior distribution on the measurand

$$p(y \mid x_1, \ldots, x_N) = \int p(y, b_1, \ldots, b_N \mid x_1, \ldots, x_N)\, \mathrm{d}b_1 \cdots \mathrm{d}b_N$$

- The posterior distribution on the biases

$$p(b_i \mid x_1, \ldots, x_N) = \int p(y, b_1, \ldots, b_N \mid x_1, \ldots, x_N)\, \mathrm{d}y\, \mathrm{d}b_1 \cdots \mathrm{d}b_{i-1}\, \mathrm{d}b_{i+1} \cdots \mathrm{d}b_N$$

Other interpretations of the DoEs.
Bayesian inference for the lab's bias (normal case) .

- The model

$$X_i = X + B_i$$

- The prior

$$p(x) \propto 1;$$

$$p(b_1,\ldots,b_n | u_{B1},\ldots,u_{Bn}) = \prod_1^n \frac{1}{\sqrt{2\pi}\, u_{Bi}} \exp\left\{-\frac{b_i^2}{2u_{Bi}^2}\right\}$$

- Data

$$x_i; \quad u_{Bi}, \sigma_i \qquad x_i \in N(X + B_i, \sigma_i^2)$$

- The posterior

$$p(x, b_1,\ldots,b_n | \ldots, x_i, , u_{Bi} \ldots \sigma_i \ldots) \propto \prod_i \exp\left\{-\frac{(x_i - x - b_i)^2}{2\sigma_i^2} - \frac{b_i^2}{2u_{Bi}^2}\right\} p(x)$$

Bayesian inference for the lab's bias

- The best estimate of the measurand

$$x_w = u^2(x_w)\sum_{i=1}^{N}\frac{x_i}{(u_{Bi}^2+\sigma_i^2)}$$

$$u^2(x_w) = \left(\sum_{i=1}^{N}\frac{1}{(u_{Bi}^2+\sigma_i^2)}\right)^{-1}$$

- and the labs biases

$$\hat{b}_i = \frac{u_{Bi}^2}{u_{Bi}^2+\sigma_i^2}(x_i - x_w)$$

$$u^2\left(\hat{b}_i\right) = u_{Bi}^2\,\frac{\sigma_i^2 + u_{Bi}^2\dfrac{u^2\left(x_w\right)}{\sigma_i^2+u_{Bi}^2}}{\sigma_i^2+u_{Bi}^2}$$

$$\Rightarrow$$

$$\hat{b}_i \approx x_i - x_{ref} = d_i$$

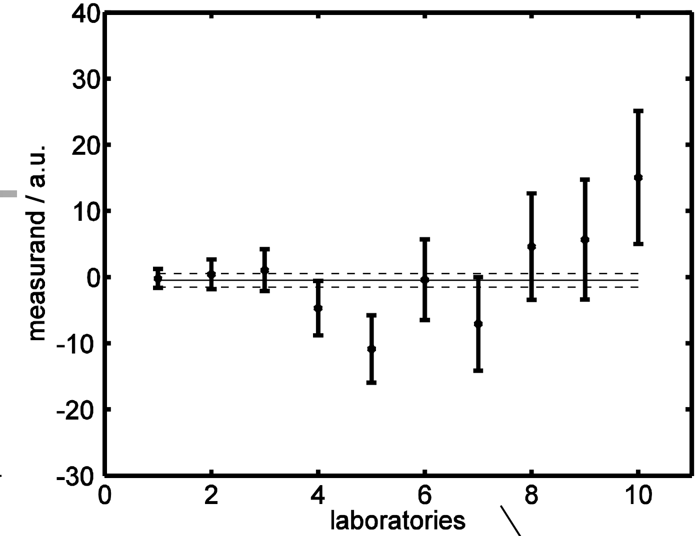
$$u^2\left(\hat{b}_i\right) \approx \sigma_i^2 + u^2\left(x_{ref}\right) \quad if \quad \sigma_i << u_{Bi}$$

Example.

- The artificial example data

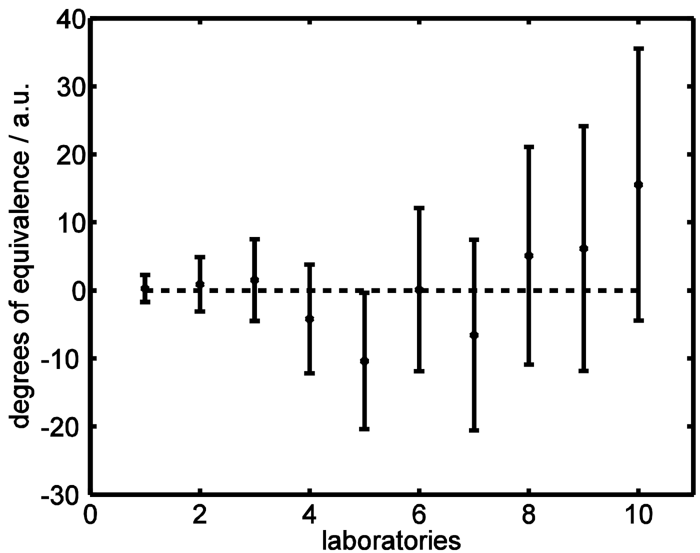
$$x_i \in N(0,u_{b,i}^2) + N(0,\sigma_i^2)$$

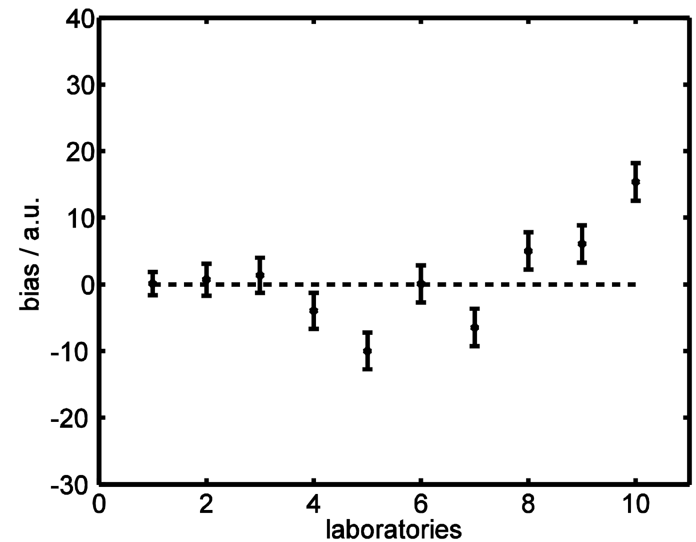
$$\sigma_i = \sigma_0 \quad u_{b,i} = l\sigma_0 \quad l = 1,\ldots,10$$



MRA approach

Laboratories' biases estimates





Summary

- an analysis of key comparison data has been proposed for the situation where detailed uncertainty information is available. The applied model and the way the information is accounted for are based on Bayesian analysis.
- estimates of the laboratories biases can result with significantly reduced uncertainties as compared with the assessment of the particular laboratory
- simple formulae were given for the case of Gaussian PDFs