Impact of Correlation on the Results of a Dynamic Calibration

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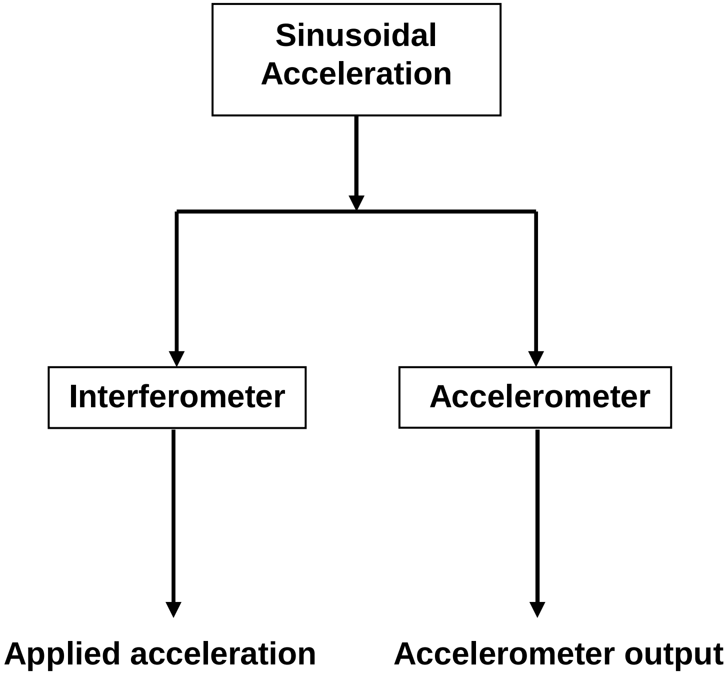
Content

- Analysis of dynamic calibration of accelerometers
- Modeling of correlation
- Impact of correlation on calibration result

Dynamic calibration of accelerometers

Sinusoidal acceleration standard at PTB
(10 Hz to 20 kHz, 10 m/s^2 up to 100 m/s^2)

(PTB Working Group 1.31)



Sinusoidal Acceleration

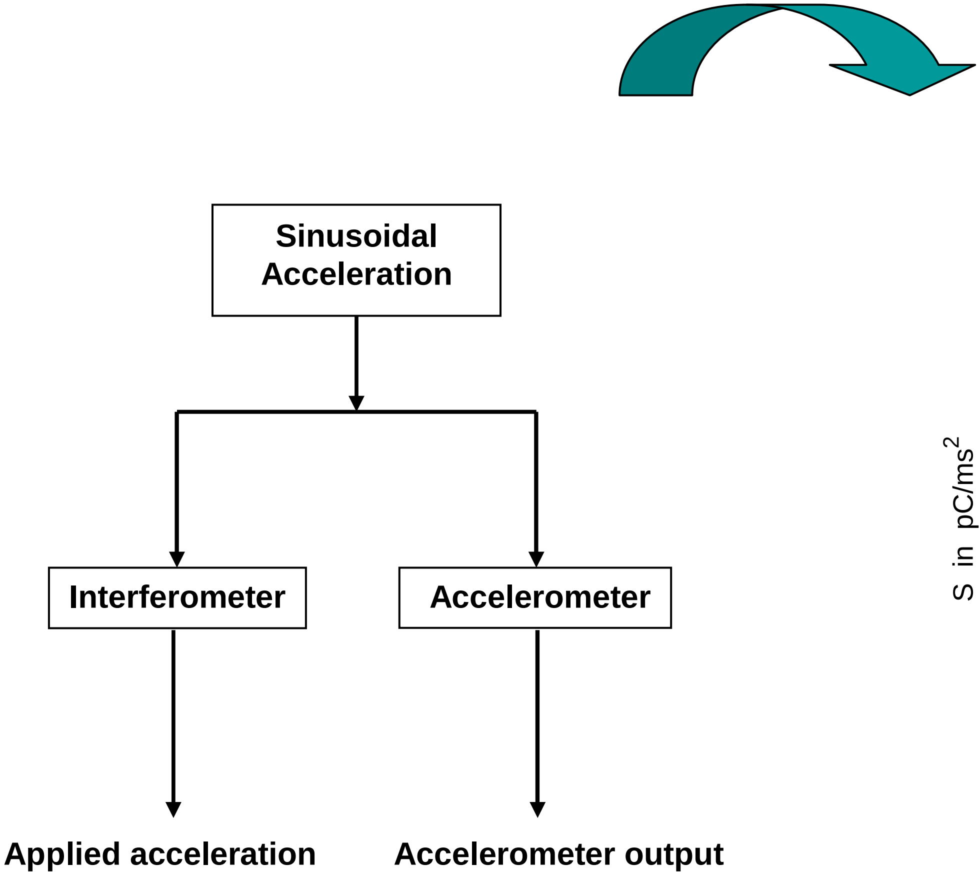
Interferometer

Accelerometer

Applied acceleration

Accelerometer output

Dynamic calibration of accelerometers



Sinusoidal Acceleration

Interferometer

Accelerometer

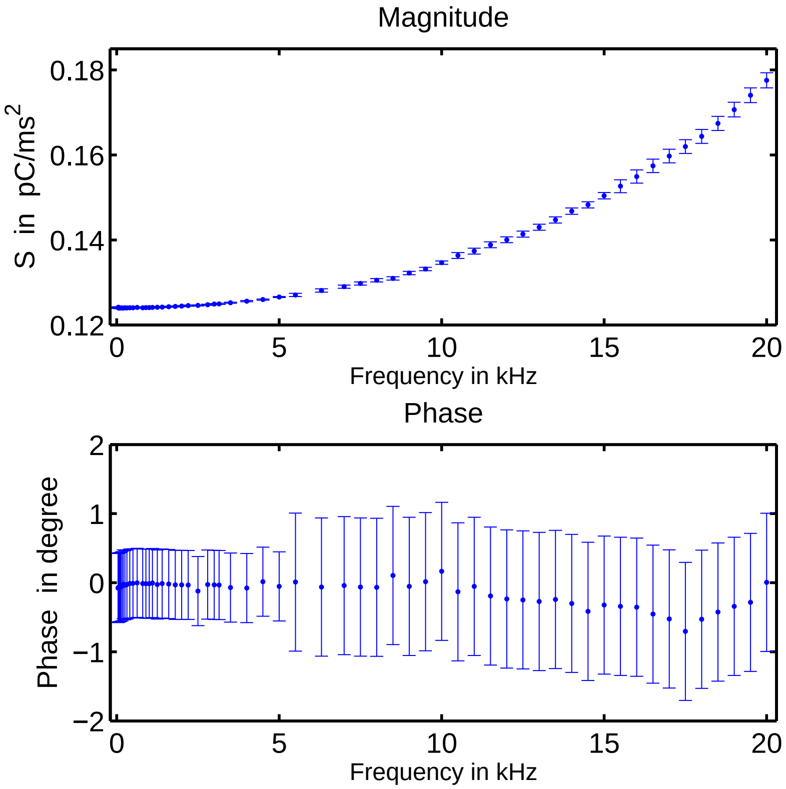
Applied acceleration

Accelerometer output

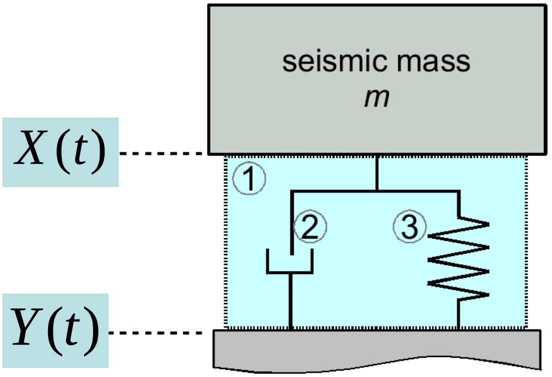
Frequency response

(complex-valued)

$$\tilde{G}(\omega) = S(\omega)e^{j\phi(\omega)}$$



Analysis of dynamic calibration



Second-order ODE

$$\ddot{X} + 2\Delta\Omega_0\dot{X} + \Omega_0^2 X = \Lambda Y$$

Frequency response

$$\tilde{G}(\omega) = \frac{\tilde{X}(\omega)}{\tilde{Y}(\omega)} = \frac{\Lambda}{(\Omega_0^2 - \omega^2) + 2j\Delta\Omega_0\omega}$$

Physical parameters

$$\Theta = \begin{pmatrix} \Delta \\ \Omega_0 \\ \Lambda \end{pmatrix}$$

Damping

Resonance frequency

Transformation-factor

Multivariate quantity

Analysis of dynamic calibration

Magnitudes $s(\omega_i),\ u(s(\omega_i))$

Phases $\varphi(\omega_i),\ u(\varphi(\omega_i))$ for frequencies $\omega_i, i = 1, 2, \ldots, L$

$$\tilde{G}(\omega) = S(\omega) e^{j\phi(\omega)} = \frac{\Lambda}{(\Omega_0^2 - \omega^2) + 2 j \Delta \Omega_0 \omega}$$

Identification

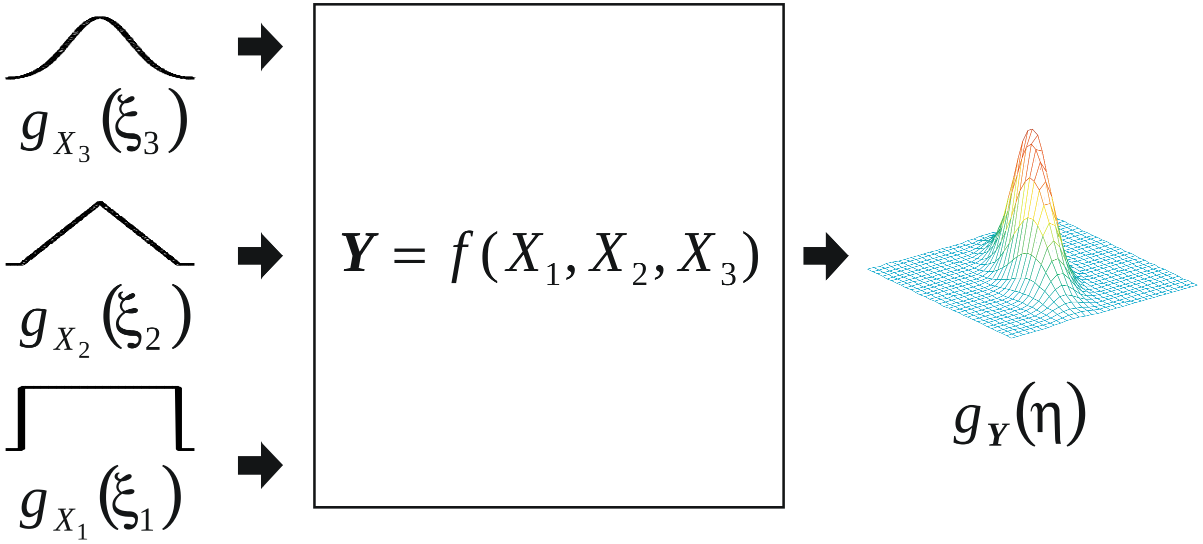
- nonlinear parameter transformation
- weighted linear least squares

Parameter estimates $\vartheta^{\mathrm{T}} = (\delta, \omega_0, \lambda)$

Uncertainty matrix $\mathbf{U}_{\vartheta}$

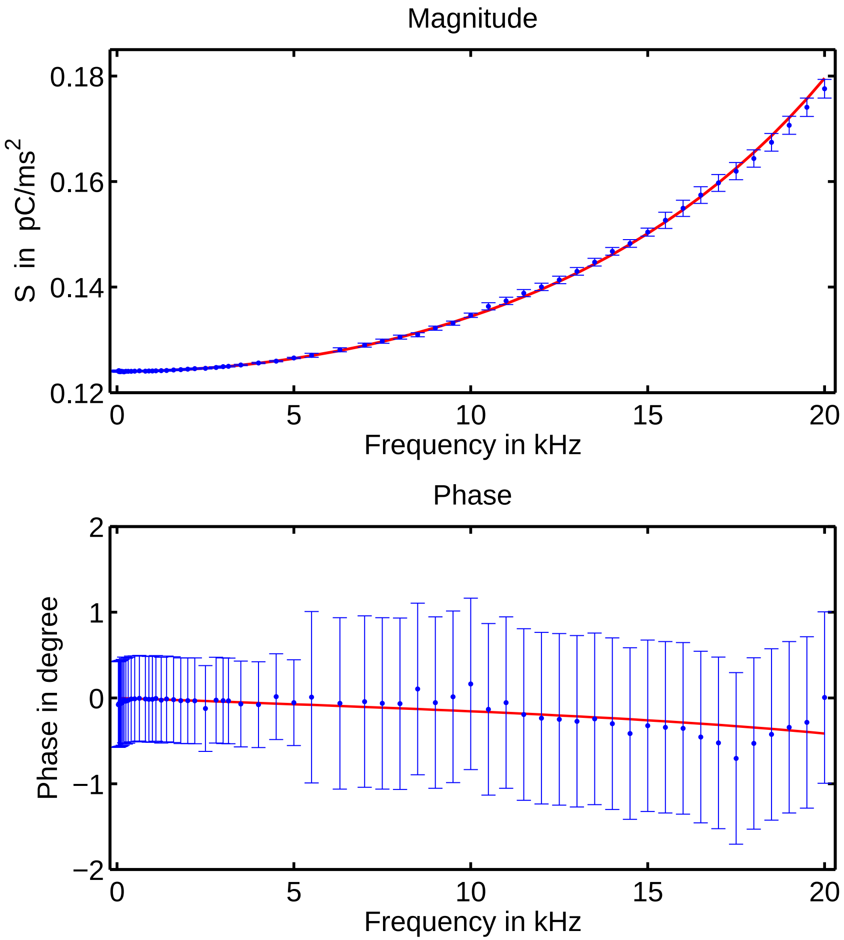
Uncertainty matrix of parameter estimates

Propagation of distributions



Uncertainty matrix $\mathbf{U}_{\vartheta}$ determined using Monte-Carlo method

Example: Dynamic calibration of an accelerometer

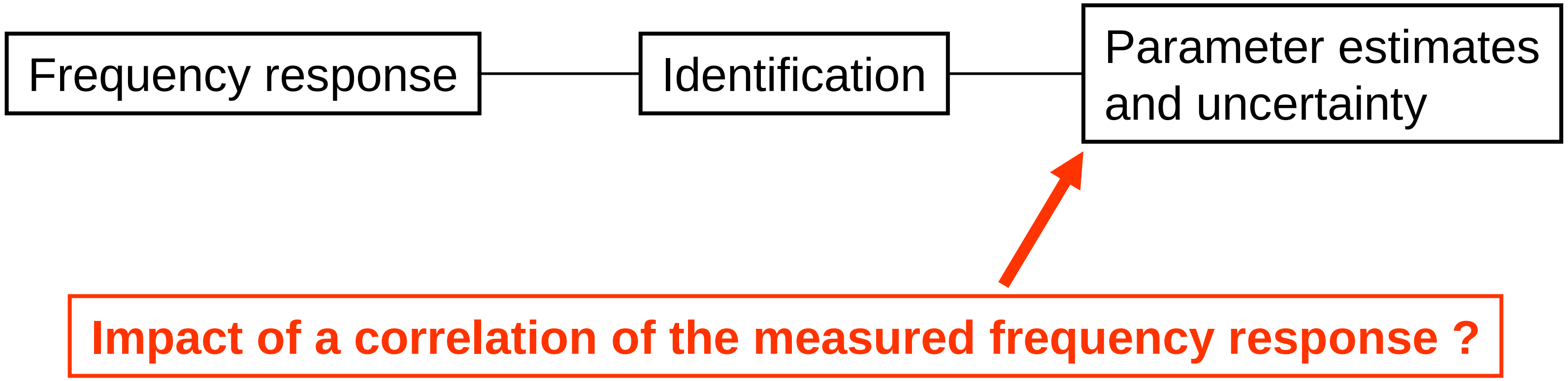


Parameter estimates

Parameter	Value	Uncertainty
δ	0.005	0.001
λ	$6.3\cdot 10^{9}$ pC/m	$1.5\cdot 10^{7}$ pC/m
f_0	35.960 kHz	0.045 kHz

$$\mathbf{U}_{\vartheta} = \begin{pmatrix} 1.43\mathrm{e}-06 & -425 & -0.0012 \\ & 2.34\mathrm{e}+14 & 6.59\mathrm{e}+08 \\ & & 1860 \end{pmatrix}$$

Modeling of correlation



Frequency response

Identification

Parameter estimates and uncertainty

Impact of a correlation of the measured frequency response ?

- **Model accounting for systematic influences**
- **Structure of correlation matrix**
- **Impact on parameter estimates ϑ and uncertainty $\mathbf{U}_{\vartheta}$**

Modeling of correlation

Vector of complex-valued frequency response $\boldsymbol{Y} = (Y_1, \ldots, Y_{2L})$

$$\left.\begin{array}{ll}\text{Magnitudes} & Y_i = S(\omega_i) \\ \text{Phases} & Y_{L+i} = \phi(\omega_i)\end{array}\right\} \text{ for frequencies } \quad \omega_i, i = 1,2,\ldots,L$$

Model

$$\begin{array}{lclcl} Y_i & = & X_i & - & \Delta_1 \\ Y_{L+i} & = & X_{L+i} & - & \Delta_2 \end{array}$$

X_i, X_{L+i} Indication of measuring system

Δ_1, Δ_2 Offset corrections

Modeling of correlation

Model

$$\begin{array}{lcccc} Y_i & = & X_i & - & \Delta_1 \\ Y_{L+i} & = & X_{L+i} & - & \Delta_2 \end{array}$$

Input estimates $\delta_1 = \delta_2 = 0$ x_i, x_{L+i}

Uncertainties $u(\delta_1),\ u(\delta_2)$ $u(x_i),\ u(x_{L+i})$

Correlations

$$\rho_1 = \frac{u(\delta_1, \delta_2)}{u(\delta_1)u(\delta_2)}$$

$$\rho_2 = \frac{u(x_i, x_{L+i})}{u(x_i)u(x_{L+i})}$$

All remaining correlations are assumed to be zero !

Modeling of correlation

Model

$$\begin{aligned} Y_i &= X_i - \Delta_1 \\ Y_{L+i} &= X_{L+i} - \Delta_2 \end{aligned}$$

Correlation matrix

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{12}^{\mathrm{T}} & \boldsymbol{R}_{22} \end{pmatrix}$$

Magnitude-magnitude

$$[\boldsymbol{R}_{11}]_{ik} = \frac{\delta_{ik} u^2(x_i) + u^2(\delta_1)}{\sqrt{u^2(x_i) + u^2(\delta_1)}\sqrt{u^2(x_k) + u^2(\delta_1)}}$$

Phase-Phase

$$[\boldsymbol{R}_{22}]_{ik} = \frac{\delta_{ik} u^2(x_{L+i}) + u^2(\delta_2)}{\sqrt{u^2(x_{L+i}) + u^2(\delta_2)}\sqrt{u^2(x_{L+k}) + u^2(\delta_2)}}$$

Magnitude-phase

$$[\boldsymbol{R}_{12}]_{ik} = \frac{\delta_{ik} \rho_2 \, u(x_i)\, u(x_{L+k}) + \rho_1 \, u(\delta_1)\, u(\delta_2)}{\sqrt{u^2(x_i) + u^2(\delta_1)}\sqrt{u^2(x_{L+k}) + u^2(\delta_2)}}$$

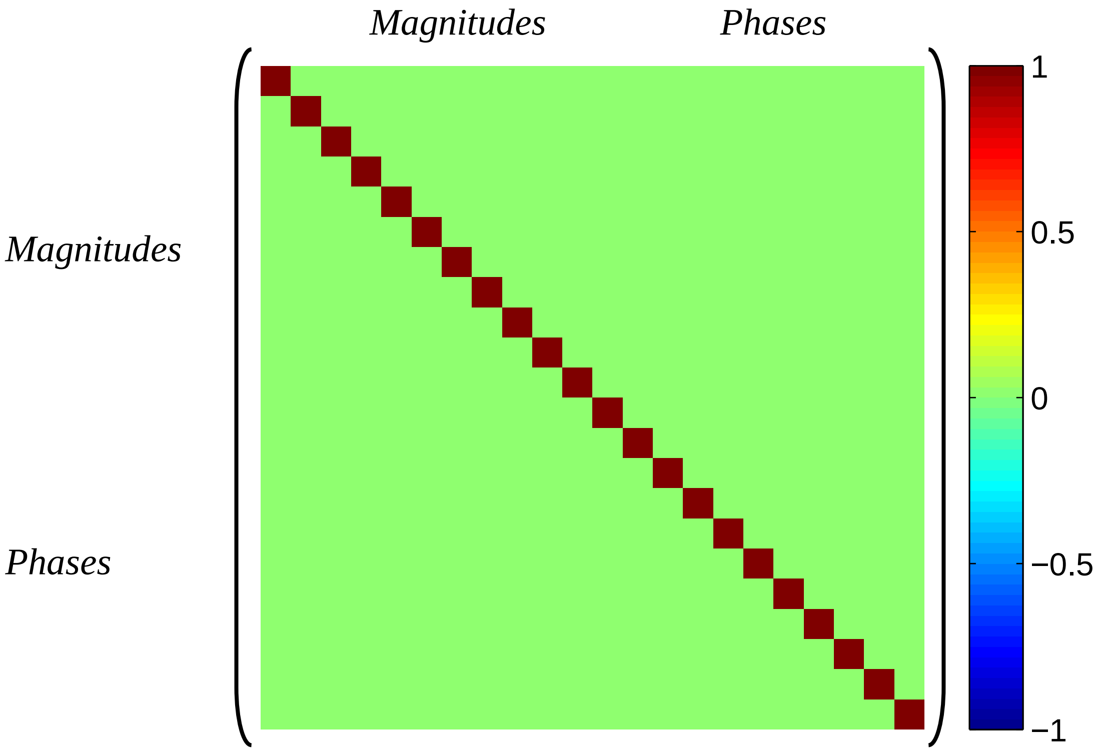
$$i = 1 \ldots L, \quad k = 1 \ldots L$$

Correlation matrix structure

$$\begin{array}{ccccc} Y_i & = & X_i & - & \Delta_1 \\ & & \rho_2 \updownarrow & & \rho_1 \updownarrow \\ Y_{L+i} & = & X_{L+i} & - & \Delta_2 \end{array}$$

$u(\delta_1)$	$u(\delta_2)$	ρ_1	ρ_2
0.0	0.0	0.0	0.0

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{12}^{\mathrm{T}} & \boldsymbol{R}_{22} \end{pmatrix}$$

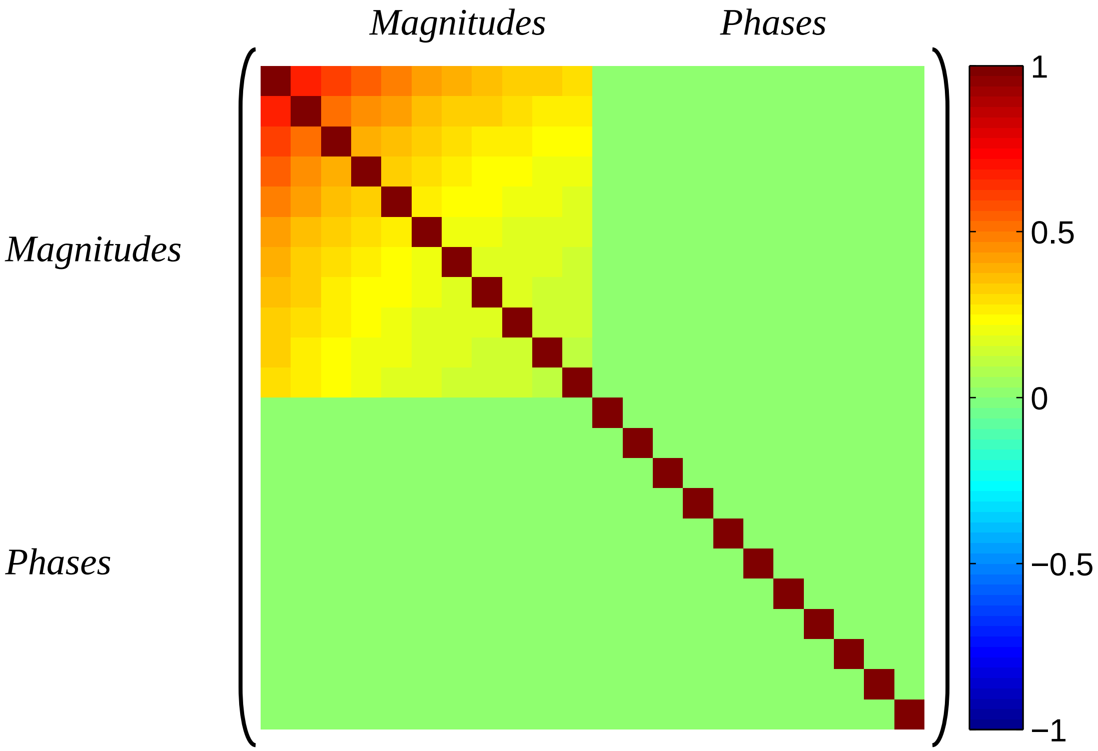


Correlation matrix structure

$$\begin{array}{lcll} Y_i & = & X_i & - \boxed{\Delta_1} \\ & & \rho_2 \updownarrow & \quad \rho_1 \updownarrow \\ Y_{L+i} & = & X_{L+i} & - \Delta_2 \end{array}$$

$u(\delta_1)$	$u(\delta_2)$	ρ_1	ρ_2
>0	0.0	0.0	0.0

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{12}^{\mathrm{T}} & \boldsymbol{R}_{22} \end{pmatrix}$$

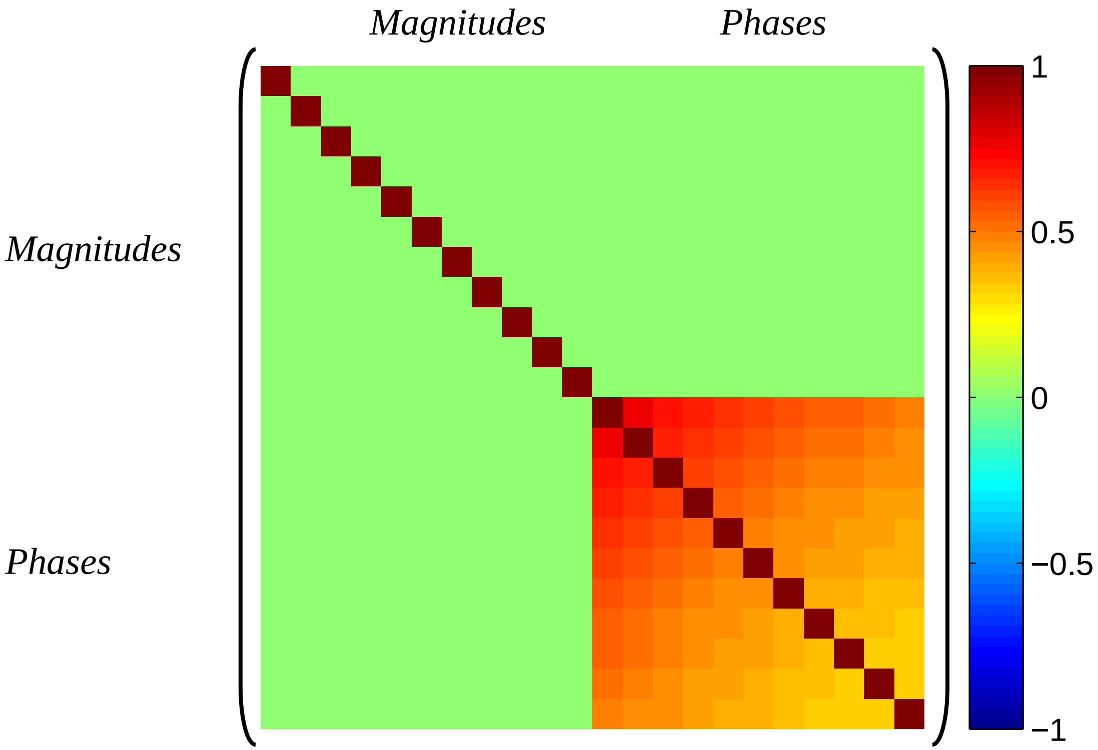


Correlation matrix structure

$$\begin{array}{ccccc} Y_i & = & X_i & - & \varDelta_1 \\ & & \rho_2 \updownarrow & & \rho_1 \updownarrow \\ Y_{L+i} & = & X_{L+i} & - & \varDelta_2 \end{array}$$

$u(\delta_1)$	$u(\delta_2)$	ρ_1	ρ_2
0.0	**>0**	0.0	0.0

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{12}^{\mathrm{T}} & \boldsymbol{R}_{22} \end{pmatrix}$$

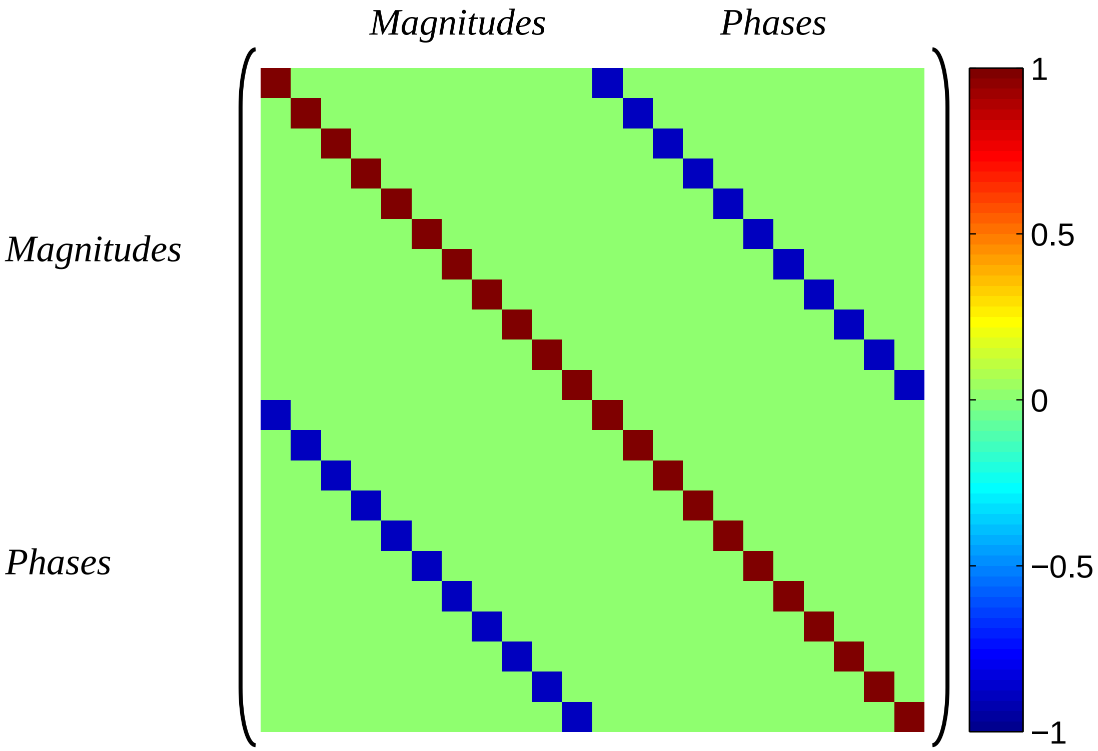


Correlation matrix structure

$$\begin{array}{ccccc} Y_i & = & X_i & - & \Delta_1 \\ & & \rho_2 \updownarrow & & \rho_1 \updownarrow \\ Y_{L+i} & = & X_{L+i} & - & \Delta_2 \end{array}$$

$u(\delta_1)$	$u(\delta_2)$	ρ_1	ρ_2
0.0	0.0	0.0	**<0**

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{12}^{\mathrm{T}} & \boldsymbol{R}_{22} \end{pmatrix}$$

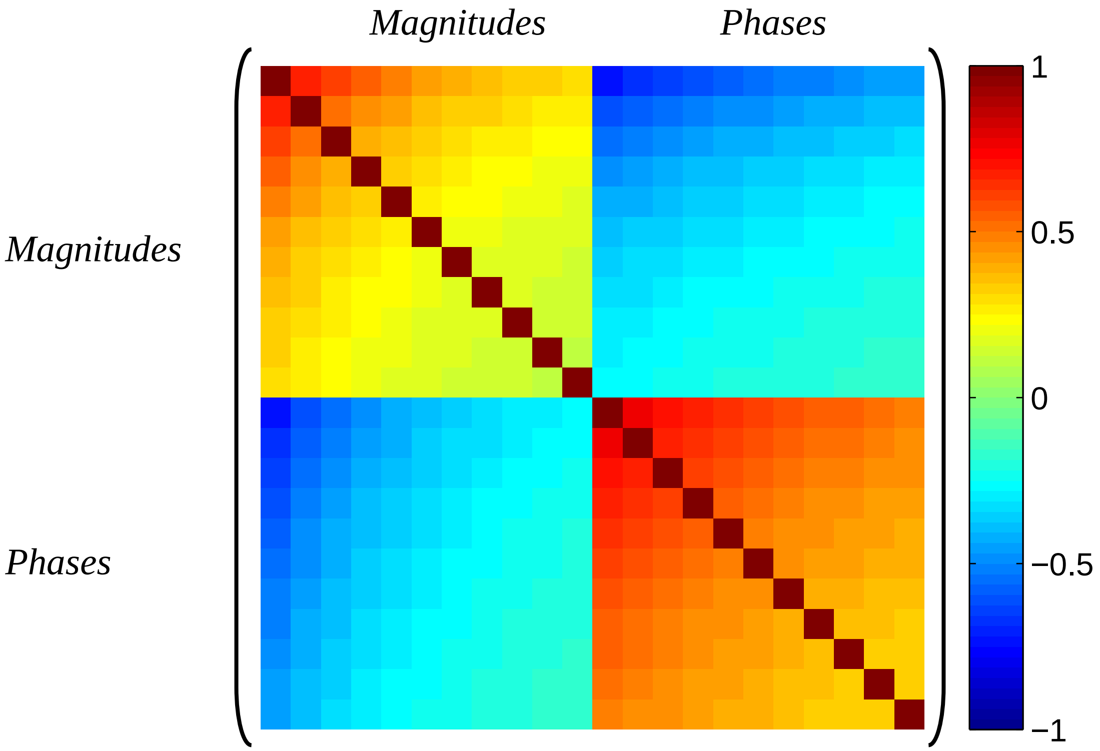


Correlation matrix structure

$$\begin{array}{ccccc} Y_i & = & X_i & - & \Delta_1 \\ & & \rho_2 \updownarrow & & \rho_1 \updownarrow \\ Y_{L+i} & = & X_{L+i} & - & \Delta_2 \end{array}$$

$u(\delta_1)$	$u(\delta_2)$	ρ_1	ρ_2
>0	**>0**	**<0**	0.0

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{12}^{\mathrm{T}} & \boldsymbol{R}_{22} \end{pmatrix}$$

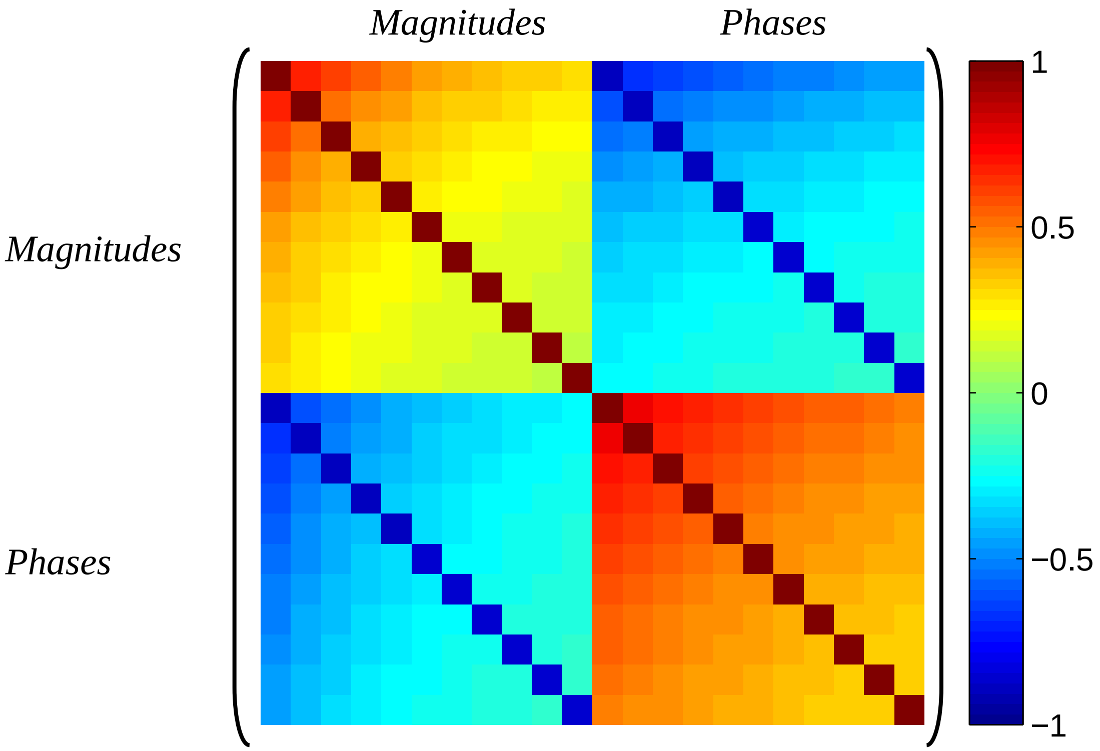


Correlation matrix structure

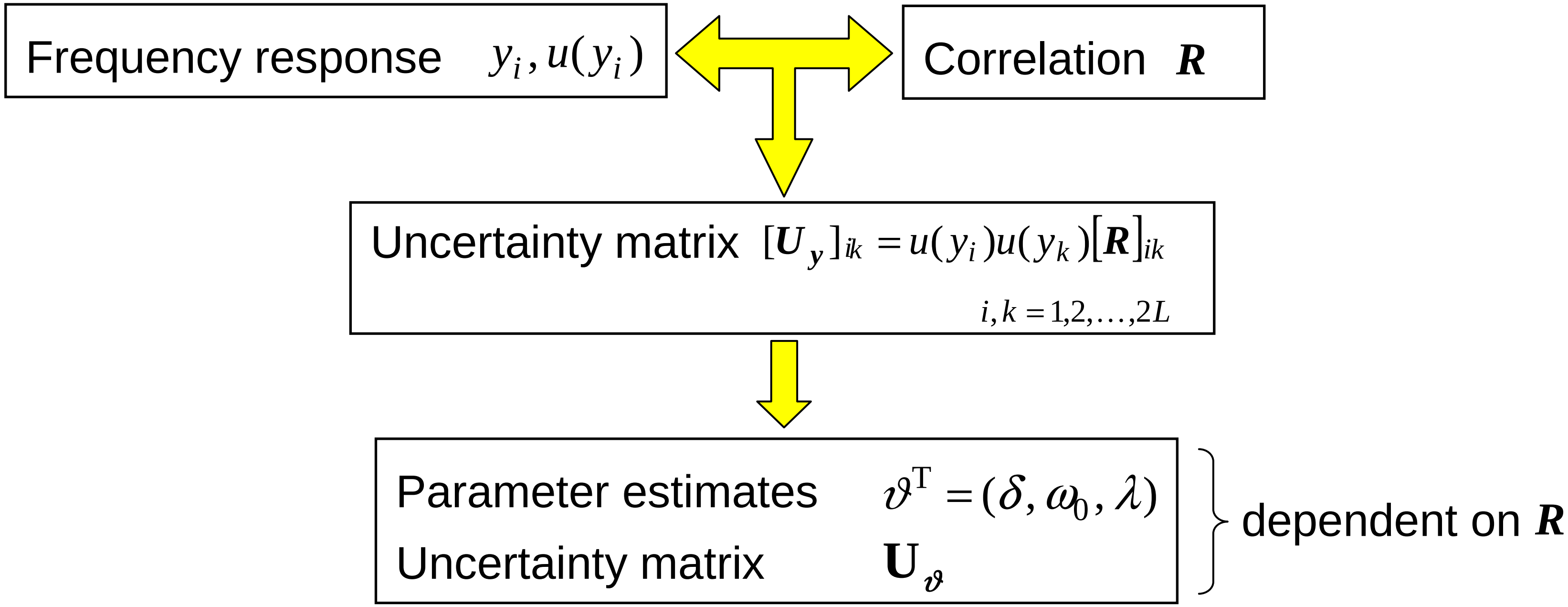
$$\begin{array}{ccccc} Y_i & = & X_i & - & \Delta_1 \\ & & \rho_2 \updownarrow & & \rho_1 \updownarrow \\ Y_{L+i} & = & X_{L+i} & - & \Delta_2 \end{array}$$

$u(\delta_1)$	$u(\delta_2)$	ρ_1	ρ_2
>0	**>0**	**<0**	**<0**

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{12}^{\mathrm{T}} & \boldsymbol{R}_{22} \end{pmatrix}$$



Impact of correlation on calibration analysis



Frequency response $y_i, u(y_i)$

Correlation $\boldsymbol{R}$

Uncertainty matrix $[\boldsymbol{U}_{\boldsymbol{y}}]_{ik} = u(y_i)u(y_k)[\boldsymbol{R}]_{ik}$

$i,k = 1,2,\ldots,2L$

Parameter estimates $\vartheta^{\mathrm{T}} = (\delta, \omega_0, \lambda)$

Uncertainty matrix $\mathbf{U}_{\vartheta}$

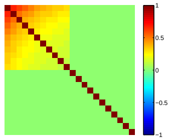
dependent on $\boldsymbol{R}$

Considered influences

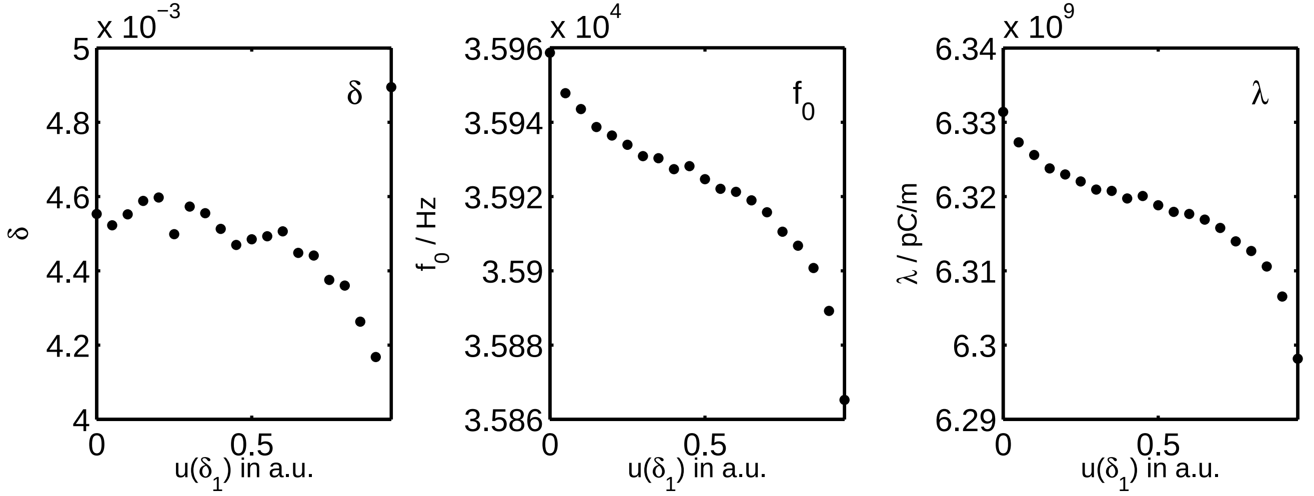
- Uncertainty of magnitude offset correction $u(\delta_1)$
- Magnitude-phase correlation ρ_2

Magnitude offset correction

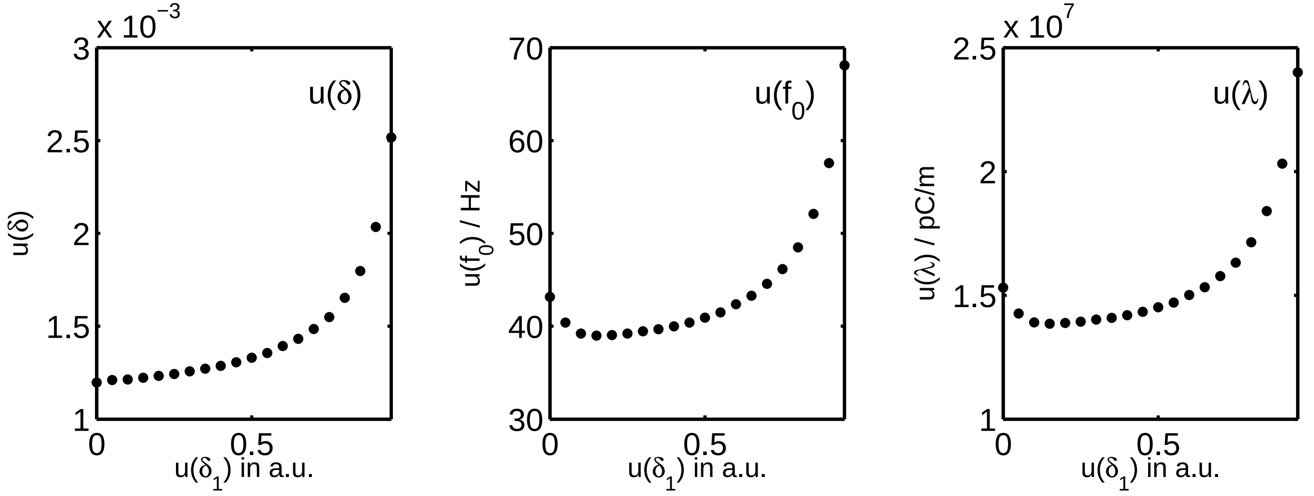
$u(\delta_1)$	$u(\delta_2)$	ρ_1	ρ_2
>0	0.0	0.0	0.0



Parameter estimates

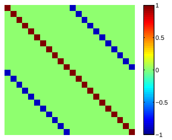


Uncertainties

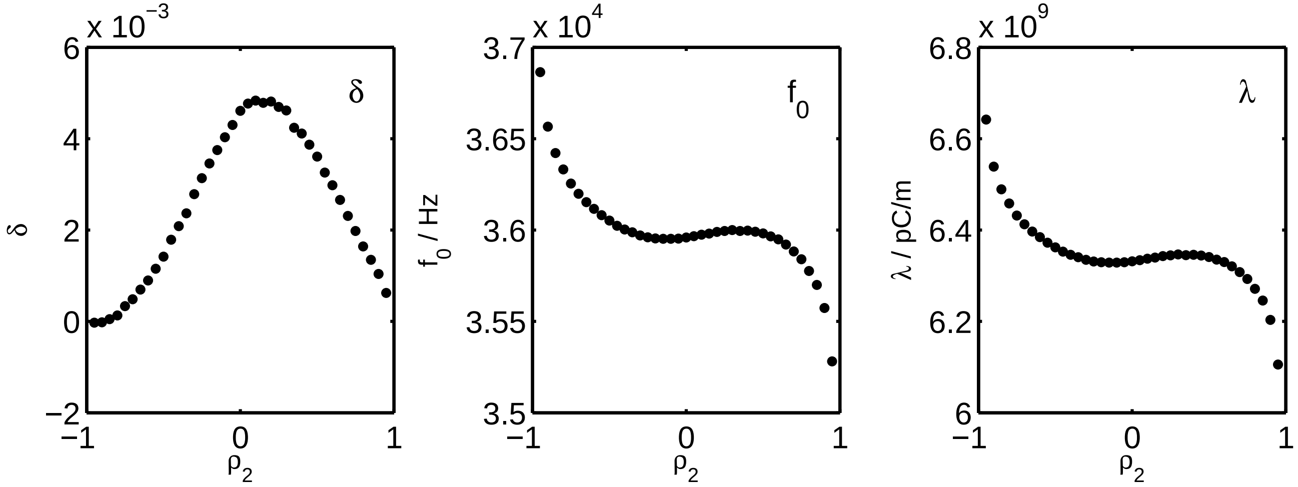


Magnitude-phase correlation

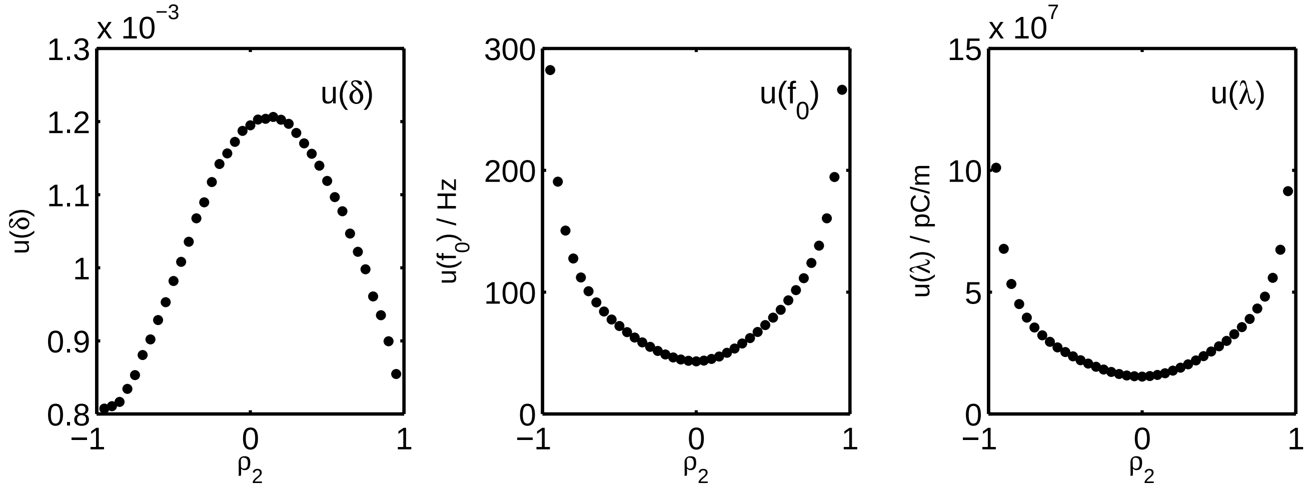
$u(\delta_1)$	$u(\delta_2)$	ρ_1	ρ_2
0.0	0.0	0.0	**[-1, 1]**



Parameter estimates



Uncertainties



Summary

- **Analysis of dynamic calibration of accelerometers**
- **Modelling systematic influences and correlation**
- **Resulting correlation structure**

Correlations may have a significant impact

on estimated parameters and uncertainties !