Possibility Distribution: a useful tool for representing coverage intervals with and without the probability density knowledge

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Plan

I- Basics of possibility theory

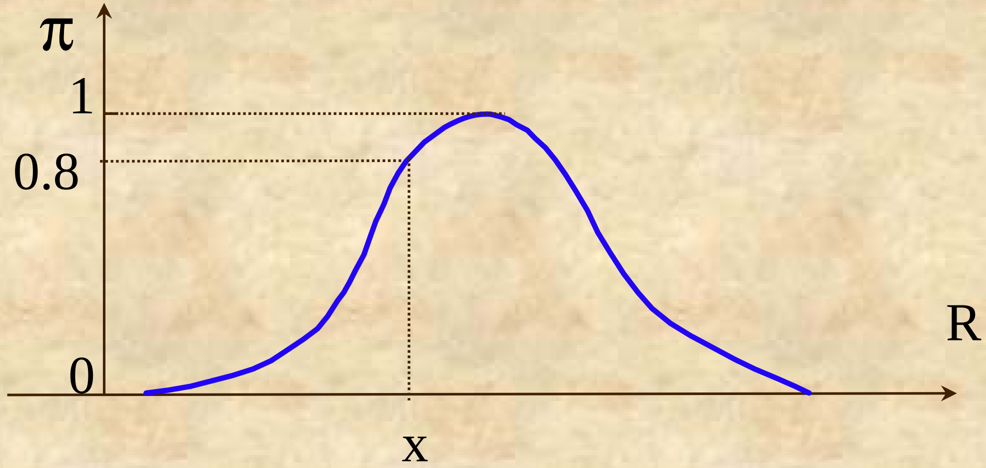
II- Probability Possibility links

III- Possibility models of incomplete probability density knowledge

IV- Conclusion/Perspectives

Possibility distribution

π: R→[0,1]



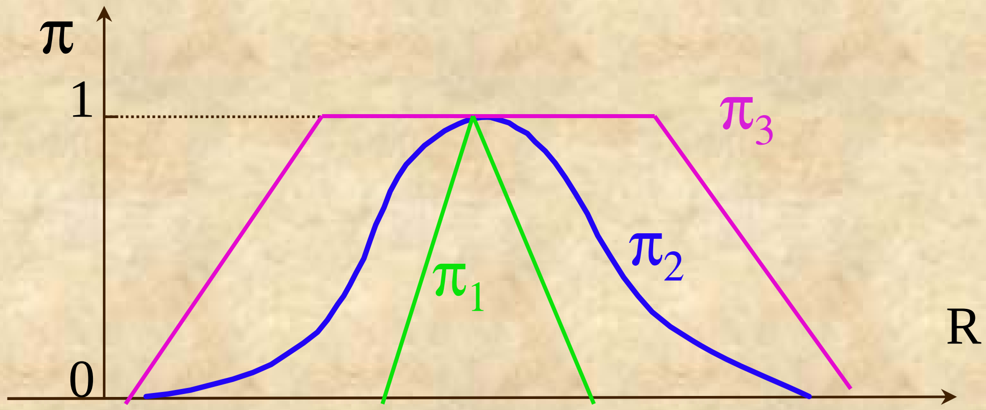
π(x) is a normalized fuzzy subset with uncertainty semantics

sup (π(x)/x belongs to R)=1(but not int(π(x)dx)=1)

π(x) represents the possibility (not the probability) that the value of the variable X is x

π(x) can be used to represent measurement uncertainty of X

Specificity of possibility distribution



For every x: $\pi_1(x) < \pi_2(x) < \pi_3(x)$

π_1 is said more specific than π_2 itself more specific than π_3

(the more specific the less spread out)

The specificity is intuitively related to the information content

Possibility Measure

$\Pi(A)=\sup(\pi(x) / x$ belongs to $A)$

$\Pi(A \cup B)=\max(\Pi(A), \Pi(B))$ (maxitivity axiom)

There exists a dual measure, the necessity measure:

$N(A)=\inf(1-\pi(x) / x$ does not belong to $A)$

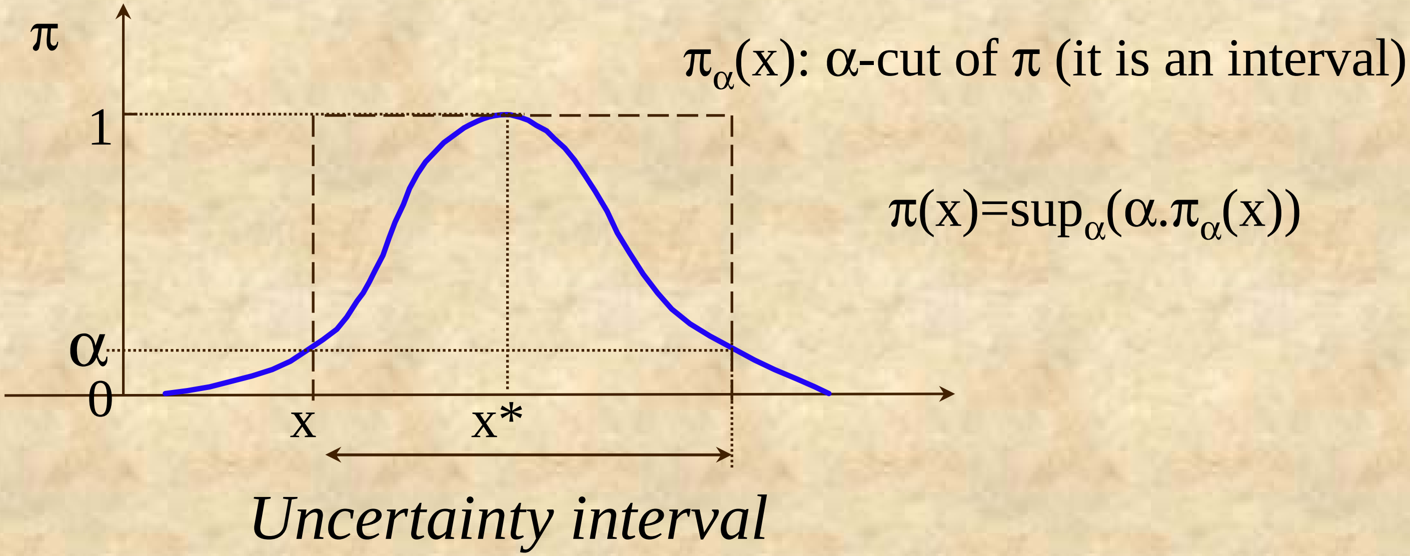
$N(A \cap B)=\min(N(A), N(B))$

$\Pi(A)=1-N(\bar{A})$

Possibility theory is similar to probability theory because it is based on set functions, but differs at a first sight by the max operation replacing the addition and the min operation replacing the product

Probability-Possibility Links

A possibility distribution can be viewed as gathering uncertainty intervals



*The α-cuts of π can be identified to the $\beta=1-\alpha$ coverage intervals of a probability density p around x**

Probability-Possibility Links

Expression of π issued from the coverage intervals of *p*

But different types of coverage intervals

$$I_\beta = \left[x_{low}(\beta), x_{high}(\beta) \right] / P(I_\beta) = \beta$$

e.g. symmetric coverage intervals around the median

$$I_\beta = \left[F^{-1}((1-\beta)/2), F^{-1}(1-(1-\beta)/2) \right]$$

$$\pi(x) = 1 - 2\left| F(x) - 0.5 \right| = \min(2F(x), 2(1-F(x))$$

π is closely related to F

(F is the c.d.f. associated to the density p)

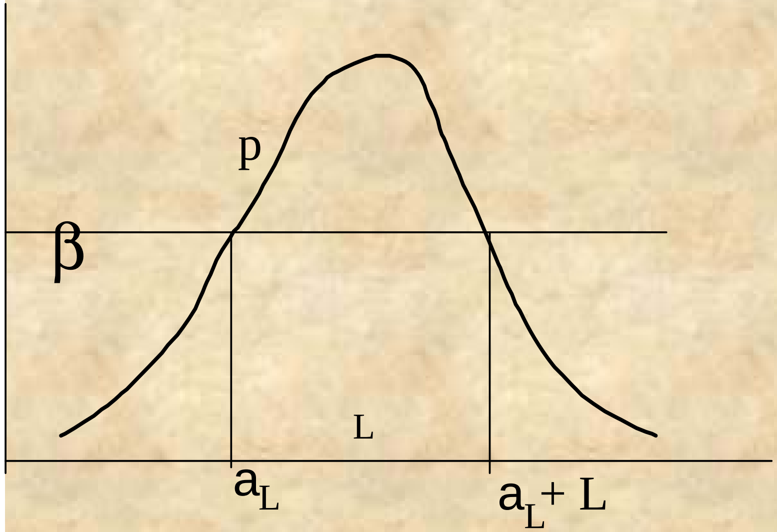
Optimal size coverage intervals

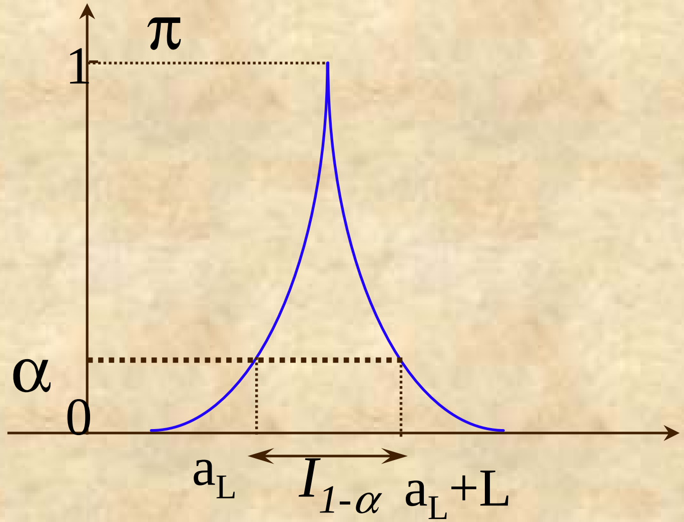
- $I_L = [a_L, a_L + L]$ is the interval of length L with maximal probability

$I_L = I_\beta = \{x, p(x) \geq \beta\}$

π^{opt} is such that $\forall L > 0$,

$\pi^{opt}(a_L) = \pi^{opt}(a_L + L) = 1 - P(I_L)$.





The α-cuts of π are identified to the β-cuts of p **with $\alpha = 1 - P(I_\beta)$**

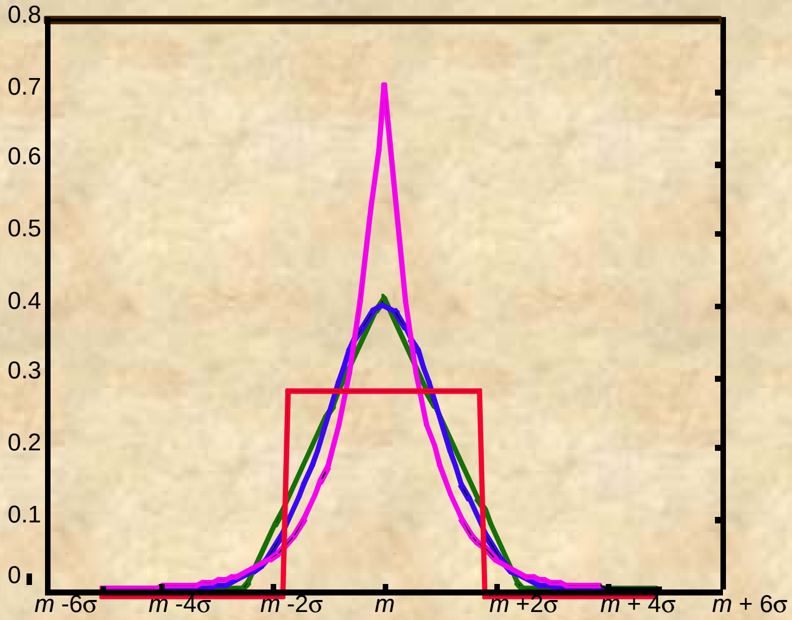
For symmetric unimodal distribution $\pi^{opt}(m-t) = \pi^{opt}(m+t) = \Pr(|X-m| \geq t)$

Symmetric unimodal distributions

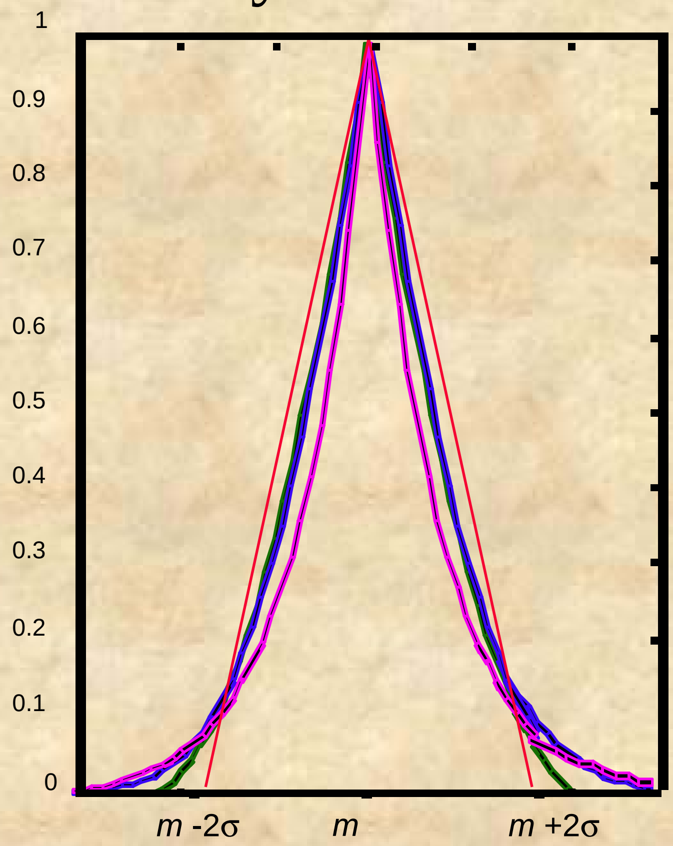
$$\forall x \in [-\infty, m], \pi^{opt}(x) = 2\int_{-\infty}^{x} p(y)dy = 2F(x)$$

$$\forall x \in [m, +\infty], \pi^{opt}(x) = 2(1 - F(x))$$

Probability distributions



Possibility distributions



In summary when p is known

It is possible to build a possibility distribution gathering all the coverage intervals of p

(ISO Guide compatibility)

It is a kind of cumulative distribution

(probability compatibility)

Intuitive interpretation

(same mode for p and π and the spread of p is translated in the specificity of π)

Possibility models of poor knowledge

Sometimes the probability density is not known, but the support, mode, standard deviation, ..., are known

In fact the available information defines a family of probability distributions F(P)

Find a possibility distribution Π dominating all probability measures P of the family Π(A)>P(A)

One solution: $\pi_{F(P)}(x) = \sup_{P \in F(P)} \pi_P^{opt}(x)$

(According to the maxitivity axiom)

Approach similar to probability inequalities

$$\pi^m(m-t) = \pi^m(m+t) = \sup_{P \in F(P)} \Pr(|X_P - m| \geq t)$$

Specificity versus entropy

- The interest is to find the maximum specific possibility distribution dominating the probability family, a simple expression is not always available but exists in some common cases (see hereafter)
- *The Maximum Entropy Principle consists in selecting a single probability distribution among the family*

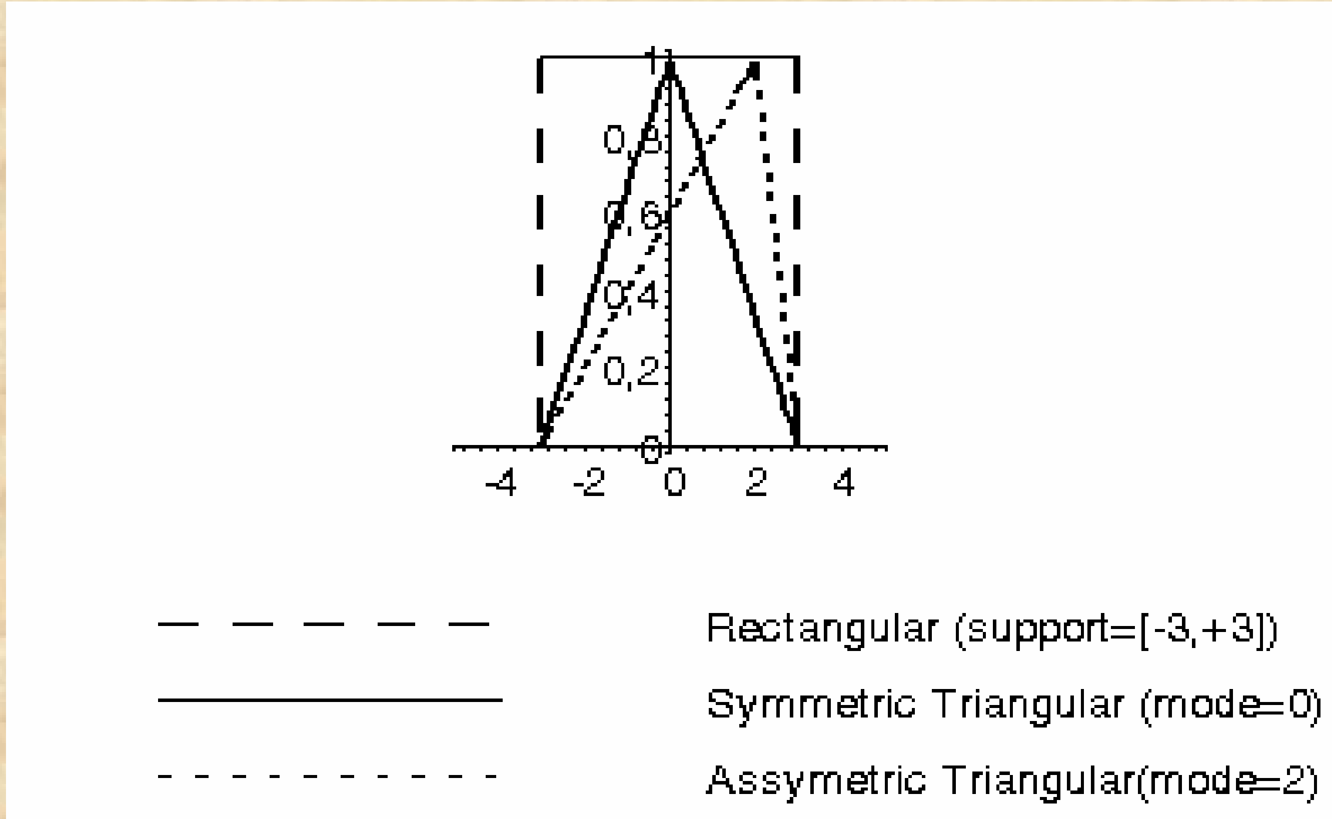
$$p_{\max ent}(x) = \arg\max_{f(x)} \left(-\int_a^b p(x)\ln(p(x))dx\right) \quad \text{Subject to} \quad \int_a^b h_i(x)p(x)dx = \mu_i$$

- The possibility approach consists in replacing the maximum entropy principle by a maximum specificity principle under the constraint of available information

Finite support examples

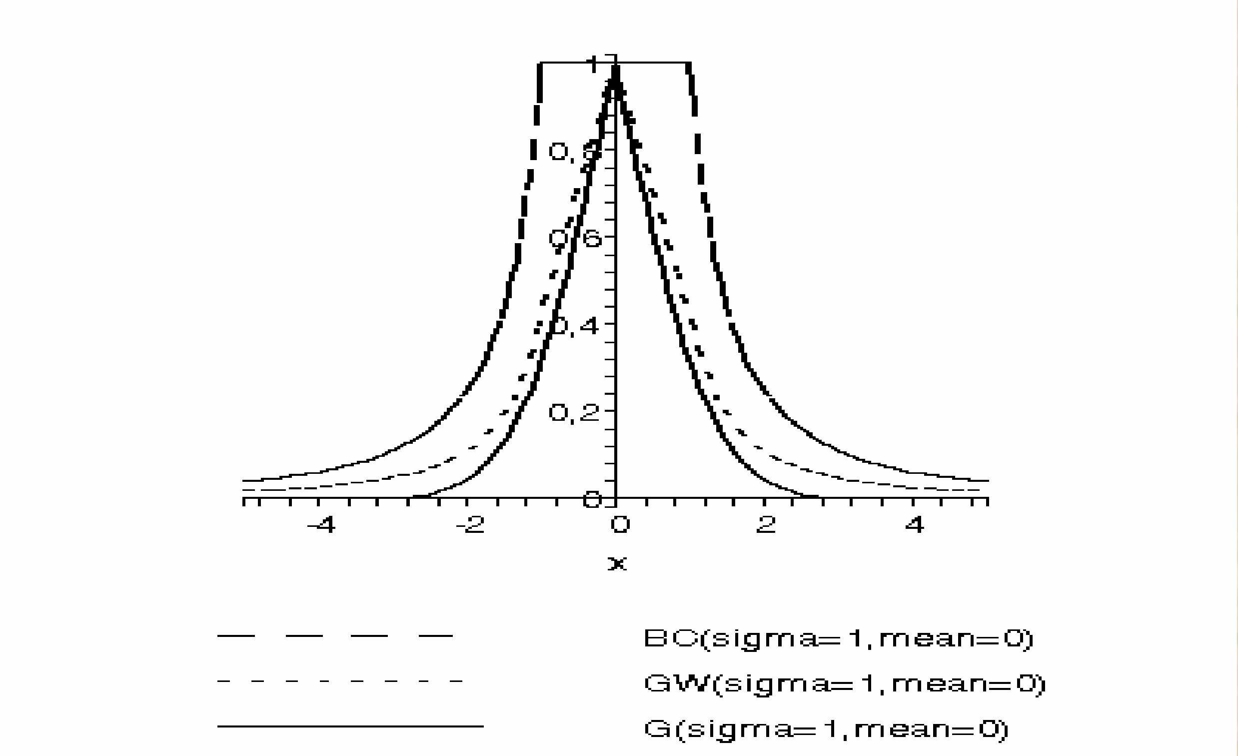
Symmetric

Asymmetric with known mode



Infinite support examples

Known mean and standard deviation
Symmetric unimodal
Gauss (the max. ent. distribution for known mean and sigma)



Conclusion

- Expression of measurement uncertainty by a possibility distribution is compatible with the ISO Guide

(*The* α*-cut of* π *can be seen as coverage intervals of* $1\text{-}\alpha$ *confidence for the known measurement probability density)*

- Simple analytical possibility distributions for some common cases of incomplete probabilistic knowledge have been obtained

- Maximum specificity principle instead of maximum entropy principle

Ceteris incognitis is more founded than ceteris paribus

Perspectives

Introducing further information
(e.g. higher moments)

Determining the best information to acquire to improve the more the specificity

Propagating the possibility distributions
(difficulties with the notion of dependence)