Framework for Evaluation of Uncertainty with Test of Linearity using Covering Arrays

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The Mainstream GUM Method

measurement function $$Y = f(X_1, X_2, \ldots, X_n)$$

⇩

linearized model $$y^* = y_0 + c_1 \cdot \Delta x_1 + c_2 \cdot \Delta x_2 + \ldots + c_n \cdot \Delta x_n$$

sensitivity $$c_i = \left.\frac{\partial Y}{\partial X_i}\right|_{X_1 = x_{1,0}, \ldots, X_n = x_{n,0}}$$

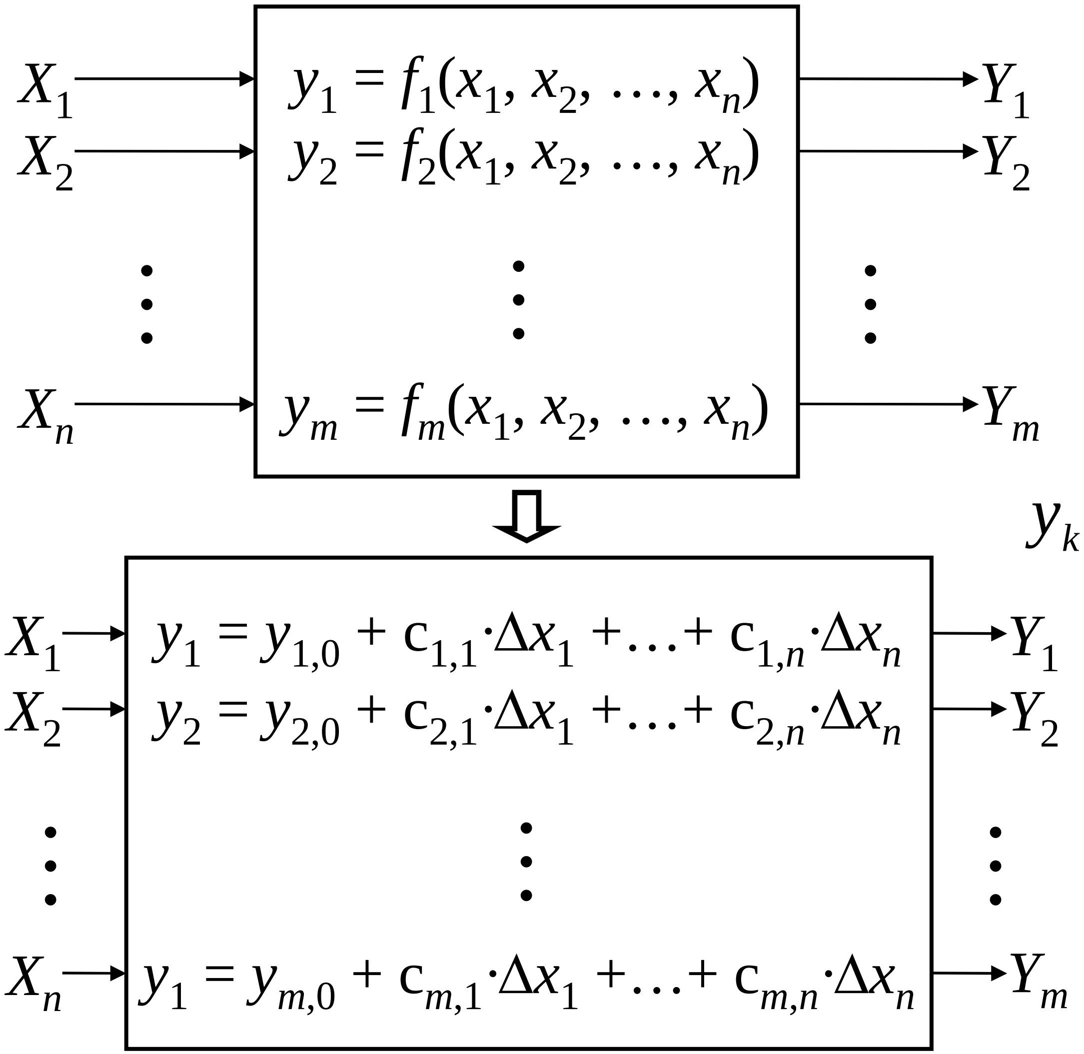
⇩

uncertainty propagation (uncorrelated inputs) $$u^2(y) = \sum_{i=1}^{n} c_i^2 u^2(x_i)$$

⇩

expanded uncertainty $$U(y) = k \cdot u(y)$$

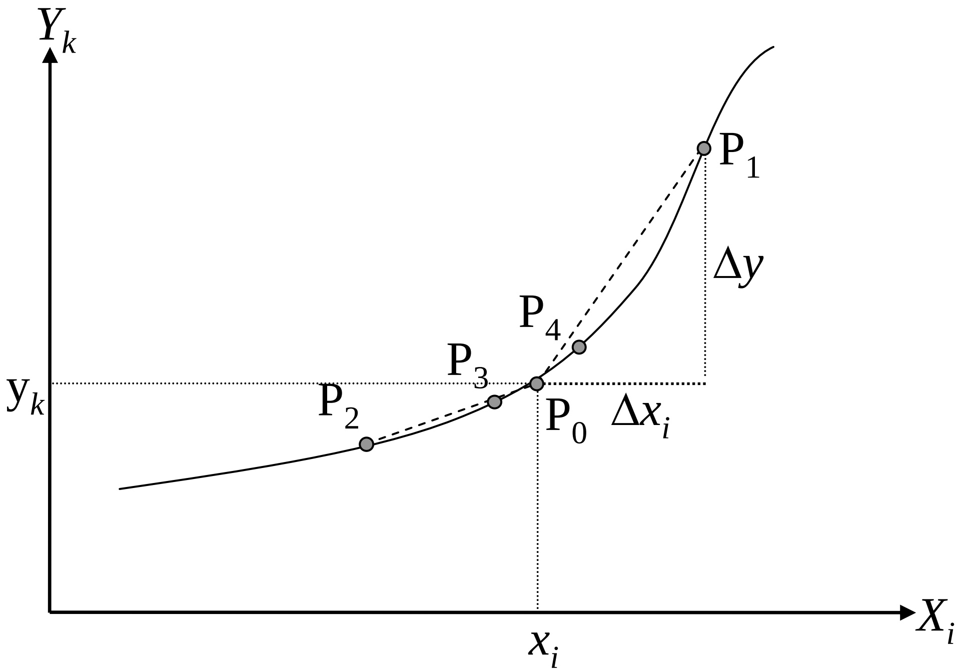
The Linearized Model



$$y_{k,0} = f_k(x_{1,0}, x_{2,0}, \ldots, x_{n,0})$$

$$\Delta x_i = x_i - x_{i,0}$$

Sensitivity Analysis



$$\frac{\partial y_k}{\partial x_i} \cong \frac{\Delta y_k}{\Delta x_i} = \frac{y_{k,1} - y_{k,0}}{x_{i,1} - x_{i,0}} = c_{k,i}$$

$$\Delta x_i = u(x_i)$$

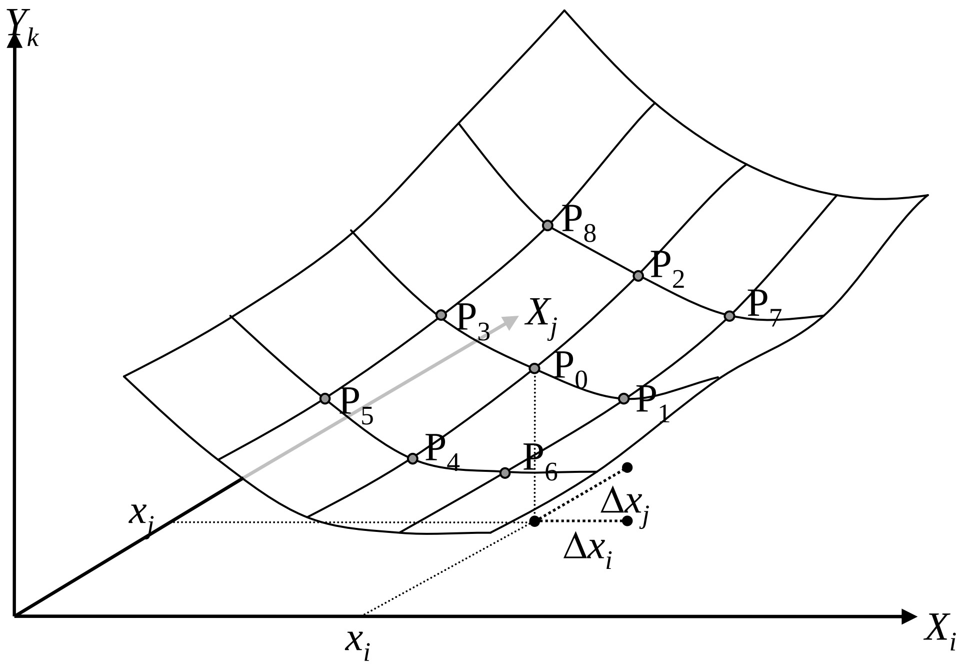
$$S_{k,i} = c_{k,i} \cdot \frac{x_i}{y_k}$$

linearity check:

$$c_{k,i}(w) = w \cdot \frac{f_k\left(\ldots, x_i + \dfrac{u(x_i)}{w}, \ldots\right) - f_k(\ldots, x_i, \ldots)}{u(x_i)}$$

$$\left|\frac{c_{k,i}(1)}{c_{k,i}(-1)} - 1\right| < \varepsilon_{\mathrm{lin}} \qquad \left|\frac{c_{k,i}(1)}{c_{k,i}(10)} - 1\right| < \varepsilon_{\mathrm{lin}} \qquad \left|\frac{c_{k,i}(-1)}{c_{k,i}(-10)} - 1\right| < \varepsilon_{\mathrm{lin}} \qquad \varepsilon_{\mathrm{lin}} = 0.05$$

Two Parameter Coupling Test



Example: $Y = X_1 \cdot X_2$

Point	w_i	w_j
P_5	–1	–1
P_6	+1	–1
P_7	+1	+1
P_8	–1	+1

$$d_k(w_i, w_j) = f_k\left(\ldots, x_i + \frac{u(x_i)}{w_i}, x_j + \frac{u(x_j)}{w_j}, \ldots\right) - f_k(\ldots, x_i, x_j, \ldots)$$

$$l_k(w_i, w_j) = c_{k,i}(1) \cdot \frac{u(x_i)}{w_i} + c_{k,j}(1) \cdot \frac{u(x_j)}{w_j}$$

$$\left| \frac{d_k(w_i, w_j) - l_k(w_i, w_j)}{l_k(w_i, w_j)} \right| < \varepsilon_{\text{lin}}$$

Efficiency of the method

- Linear model $n + 1$ model calculations
- Linearity check $(p - 2) \cdot n$ (p is 3 or 5)
- All pairs coupling test $(n^2 - n)/2$
- $n = 50$ leads to 2225 model evaluations for the coupling test

⇒ A more efficient coupling test is needed.

Solution: analyze multiple pairs in one run.

Covering Arrays

Example: 10 parameters binary covering array with strength two

run	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}
1.	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
2.	−1	+1	+1	+1	+1	+1	+1	+1	+1	+1
3.	+1	−1	+1	−1	+1	−1	+1	−1	+1	−1
4.	+1	+1	−1	+1	−1	+1	−1	+1	−1	+1
5.	−1	+1	+1	−1	−1	−1	−1	+1	+1	−1
6.	−1	−1	−1	+1	+1	+1	+1	−1	−1	+1
7.	−1	+1	+1	+1	+1	−1	−1	−1	−1	−1
6.	−1	−1	−1	−1	−1	+1	+1	+1	+1	+1
9.	+1	−1	+1	−1	−1	+1	+1	+1	−1	−1
10.	+1	−1	−1	+1	+1	−1	−1	−1	−1	+1

Generated with FireEye

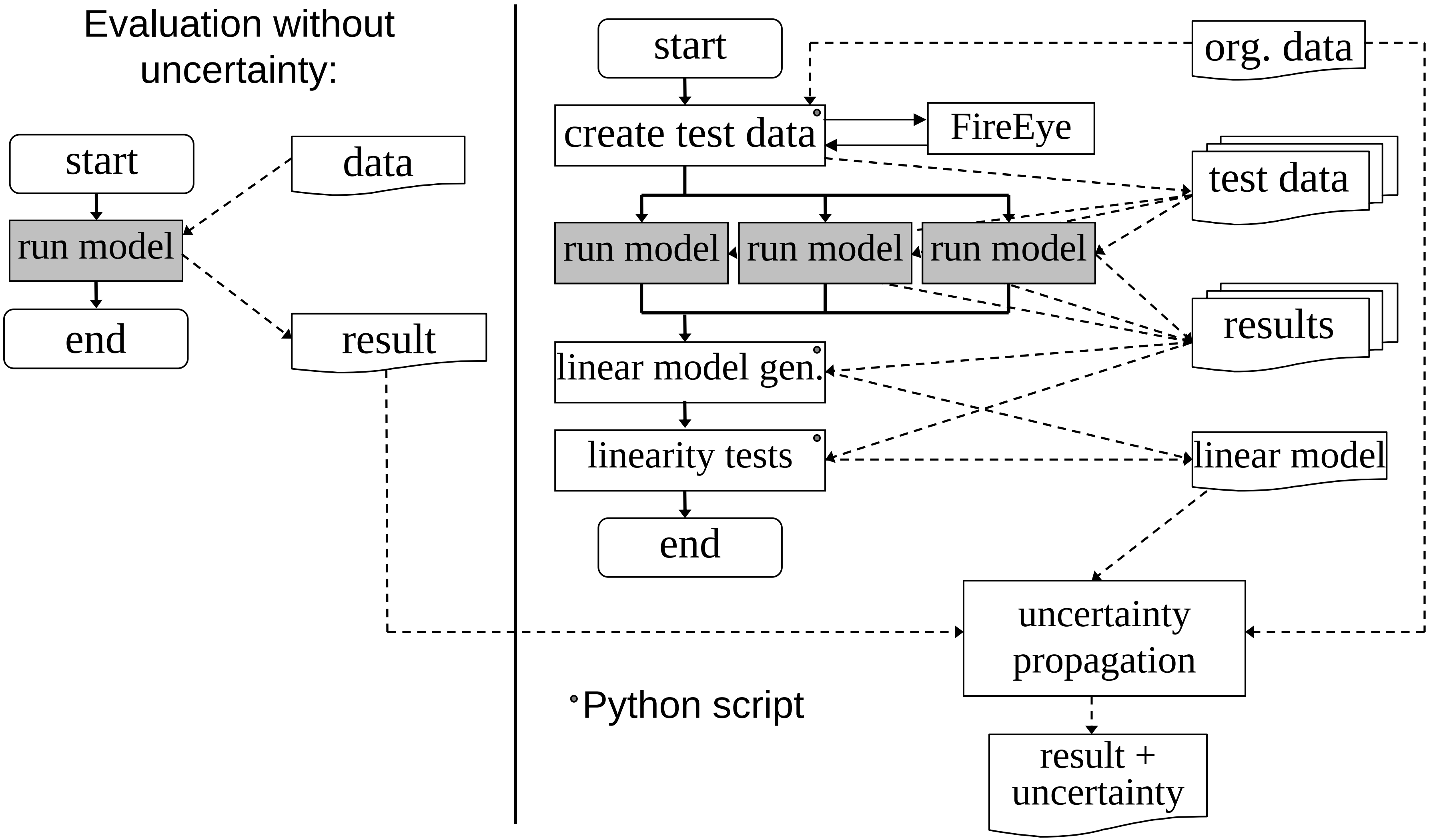
$$d_k^*(w_1,\dots,w_n) = f_k\left(x_1 + \frac{u(x_1)}{w_1},\dots,x_n + \frac{u(x_n)}{w_n}\right) - f_k(x_1,\dots,x_n)$$

$$l_k^*(w_1,\dots,w_n) = \sum_{i=1}^{n} c_{k,i}(1)\cdot\frac{u(x_i)}{w_i}$$

$$\left|\frac{d_k^*(w_1,\dots,w_n) - l_k^*(w_1,\dots,w_n)}{l_k^*(w_1,\dots,w_n)}\right| < \varepsilon_{lin}$$

Efficiency: Number of rows ~ $\log_2(n)$ for large n

Flowchart:
Linear Model Generation



Conclusion

- The GUM method can be applied to any kind of result evaluation
- The framework can be applied to command line programs
- Some useful linearity and coupling tests can be applied
- Covering Arrays define efficient test cases for the coupling test