Digital Filtering for Dynamic Uncertainty

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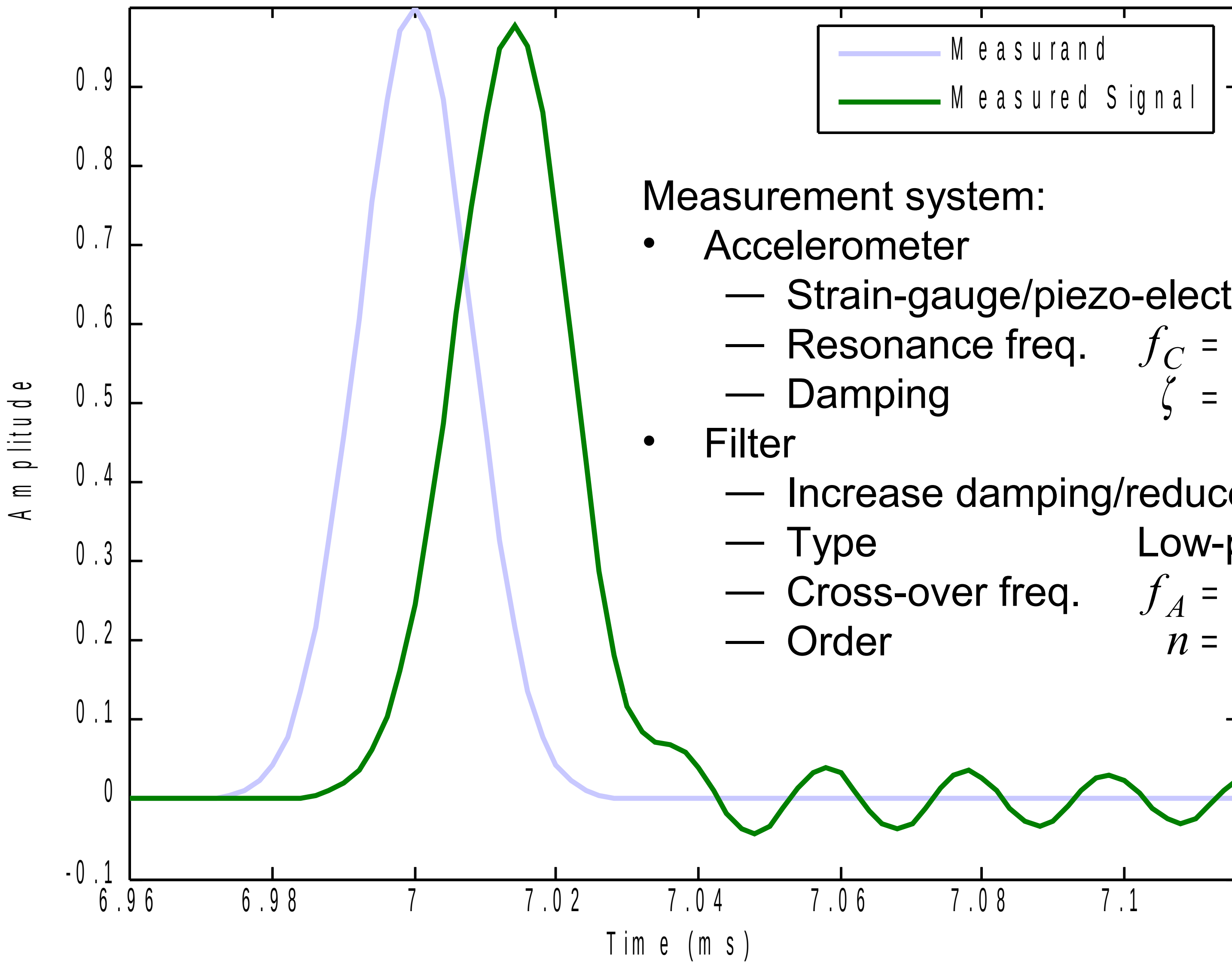
Measurement Technology

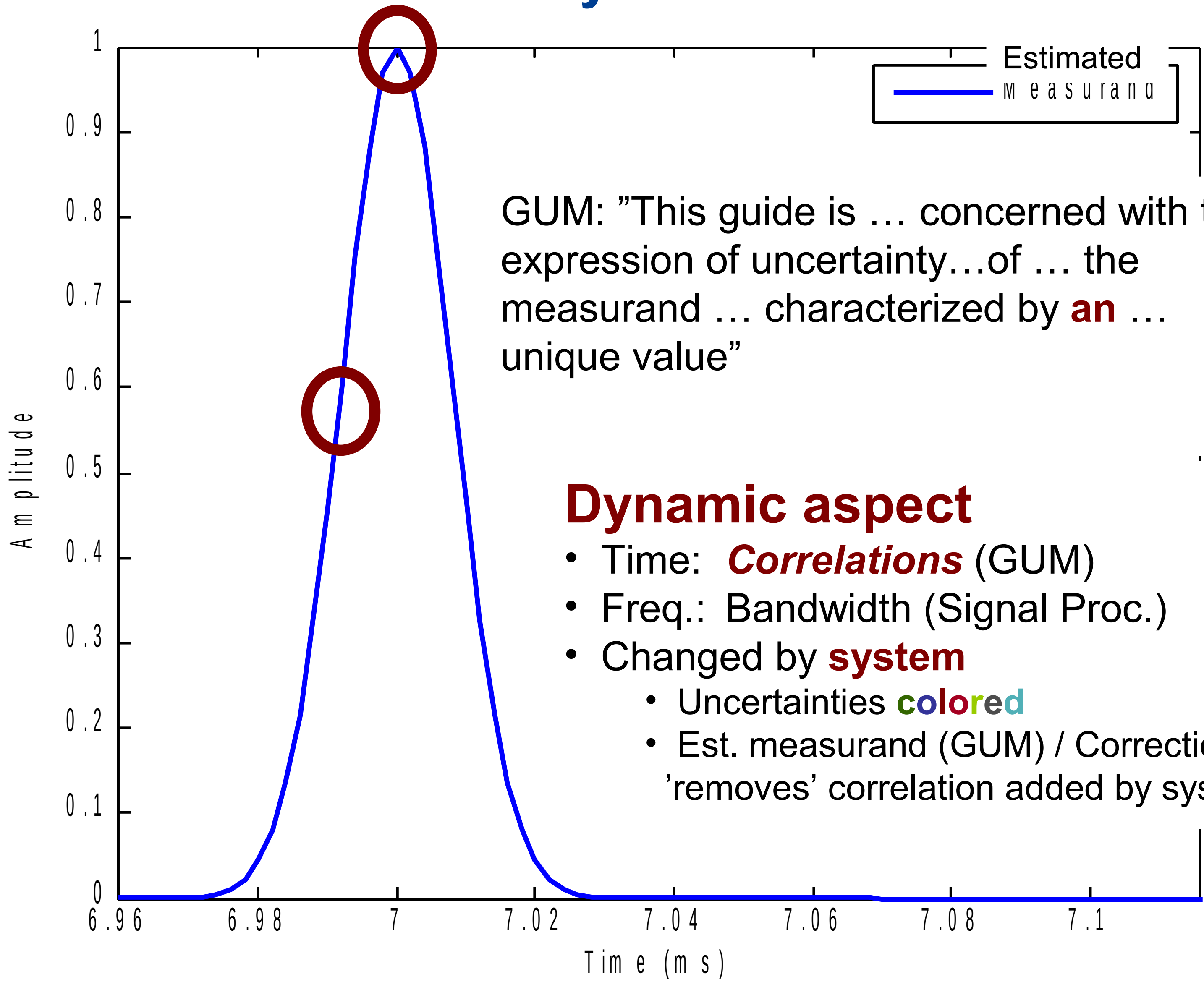
The dynamic measurement



Measurement system:

- Accelerometer
  - Strain-gauge/piezo-electric
  - Resonance freq. $f_C = 50\,\text{kHz}$
  - Damping $\zeta = 0.02$
- Filter
  - Increase damping/reduce oscillations
  - Type Low-pass Bessel
  - Cross-over freq. $f_A = 0.75 f_C$
  - Order $n = 4$

Dynamic measurand estimated !
– Measurement uncertainty ?



GUM: ”This guide is … concerned with the expression of uncertainty…of … the measurand … characterized by **an** … unique value”

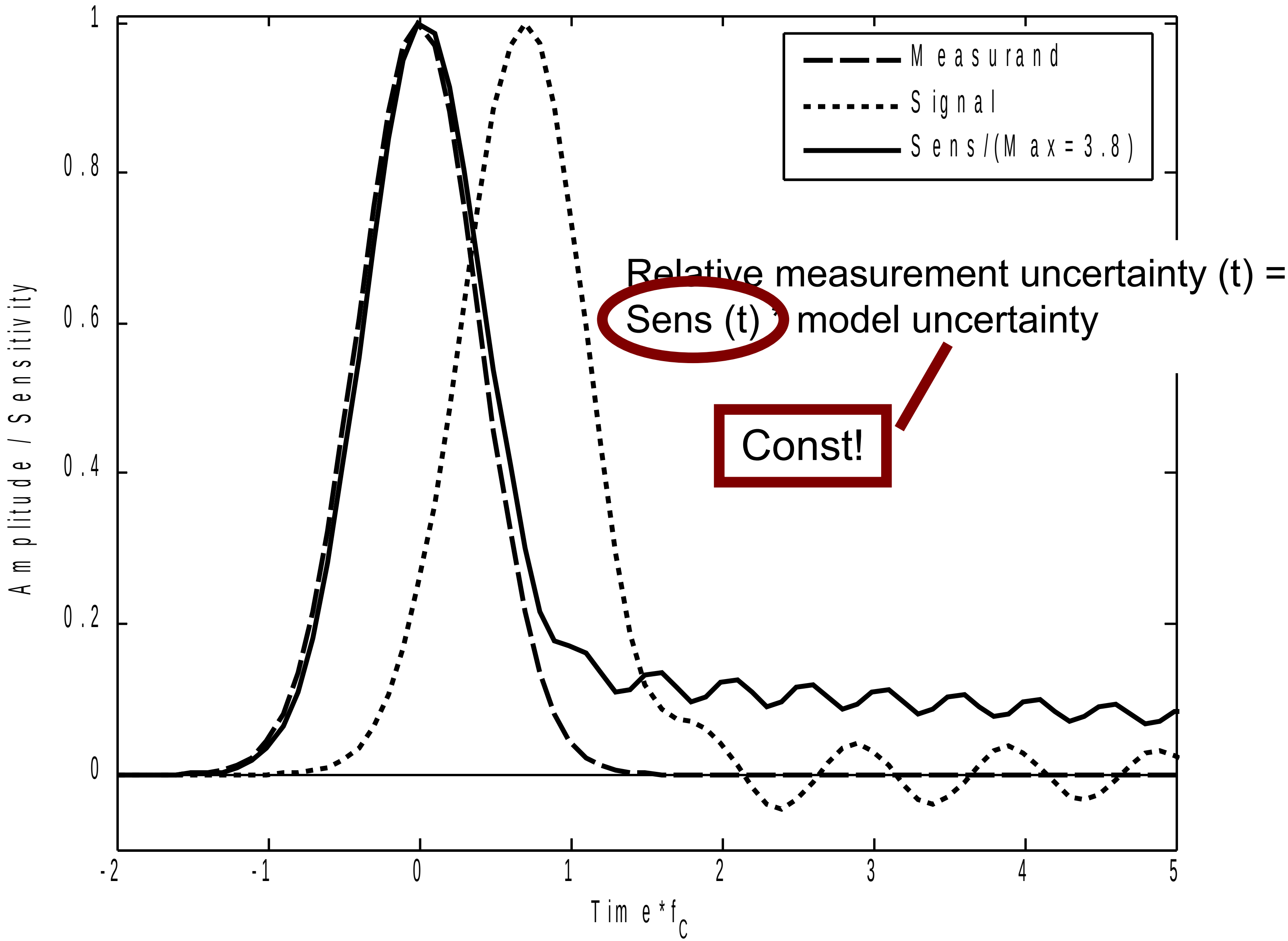
Dynamic aspect

- Time: ***Correlations*** (GUM)
- Freq.: Bandwidth (Signal Proc.)
- Changed by **system**
  - Uncertainties **colored**
  - Est. measurand (GUM) / Correction ’removes’ correlation added by system

SP

The result ?

Dynamic model uncertainty…



Outline (GUM concepts, dynamic analysis)

- Model equation for dynamic measurement (time)
- Statistics – interpretation of uncertainty
- Stochastic La-place / z-plane model
- Error model
- Error propagator
- Statistical analysis – propagator of uncertainty
- Summation rule
- Generalization of constant ’GUM’ sensitivity
- Synthesis of digital filter bank for dynamic uncertainty
- Example revisited – transducer system

Excluded:

- System identification
  - Determination of model equation from dynamic experiment / calibration
- Estimation of measurand
  - Dynamic correction
- Estimation of dynamic error

Dynamic model equation of measurement 1(2)

Continuous time CT (measurand, analogue systems)

$$\tilde{b}_0 y + \tilde{b}_1 \partial_t y + \tilde{b}_2 \partial_t^2 y + \tilde{b}_3 \partial_t^3 y + \ldots = \tilde{a}_0 x + \tilde{a}_1 \partial_t x + \tilde{a}_2 \partial_t^2 x + \tilde{a}_3 \partial_t^3 x + \ldots$$

Sampling – Analogue to Digital Conversion (ADC) ***signal*** => Aliasing

$$x_k = x(k T_S), \quad k = 1,2,\ldots \qquad f_B \leq f_N = f_S/2$$

Discrete time DT – ADC ***system*** => Discretization Time Error / Utilization

$$b_0 y_n + b_1 y_{n-1} + b_2 y_{n-2} + b_3 y_{n-3} + \ldots = a_0 x_n + a_1 x_{n-1} + a_2 x_{n-2} + a_3 x_{n-3} + \ldots$$

Measurand ('infinite' dimension) $y(t), \quad y_k$

Measured signal ('recorded') $x(t), \quad x_k$

Uncertain quantities: $b_k, \tilde{b}_k \quad a_k, \tilde{a}_k \quad y_k \quad x_k$

Dynamic model equation of measurement 2(2)

Key aspect – Differential

	CT		DT
Time:	∂_t	$\leftrightarrow$	$f_N\left(D_{+1} - D_{-1}\right)$
Transform:	s	$\leftrightarrow$	$f_N\left(z - 1/z\right)$

⇒ ***Strong correlations in time!***

- Signal
- System

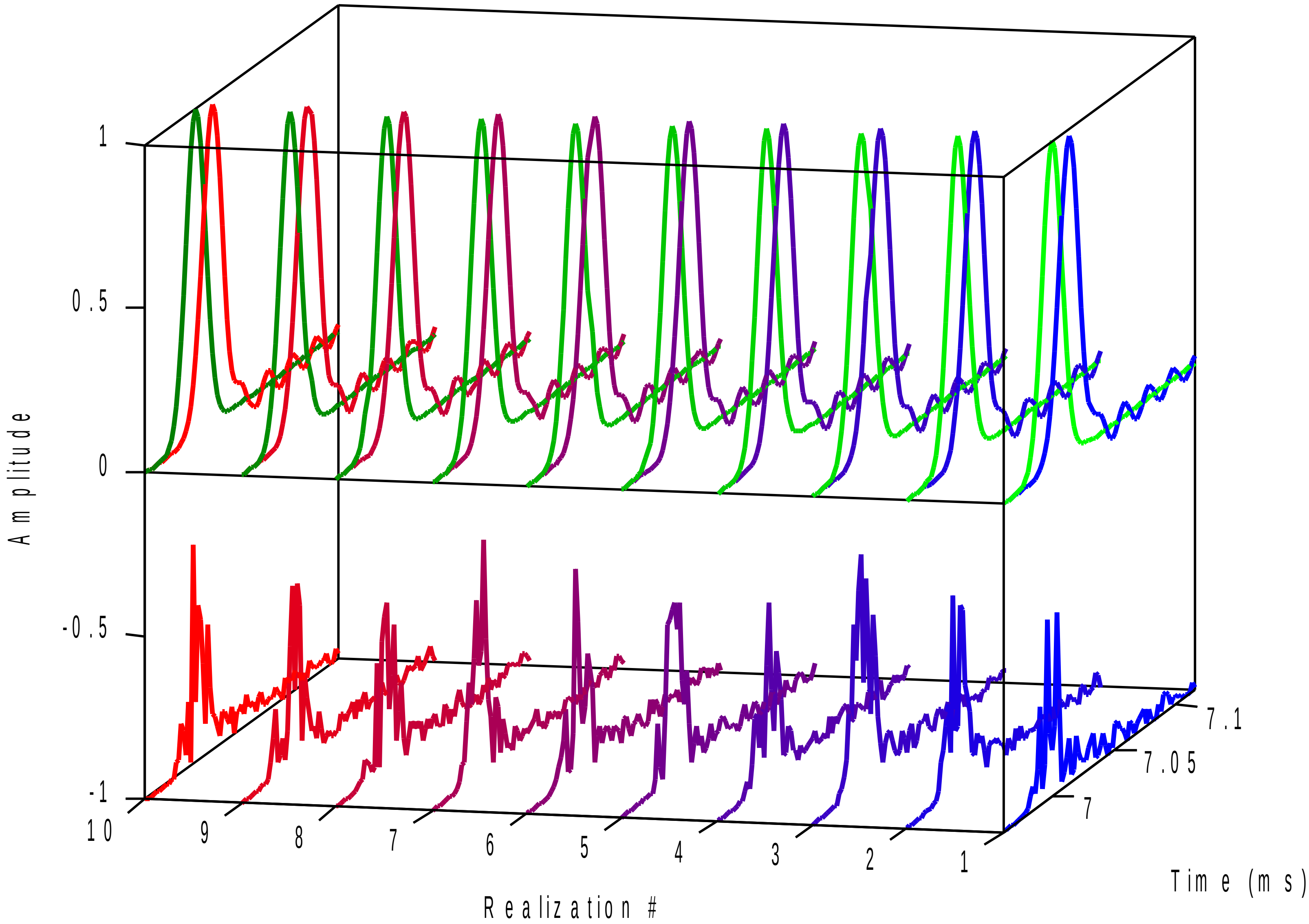
⇒ 'New'(?) features of calibration

- Waveform distortion (*auto*-correlation x / y – difference)
- Time delay (*cross*-correlation x y – lag for max)

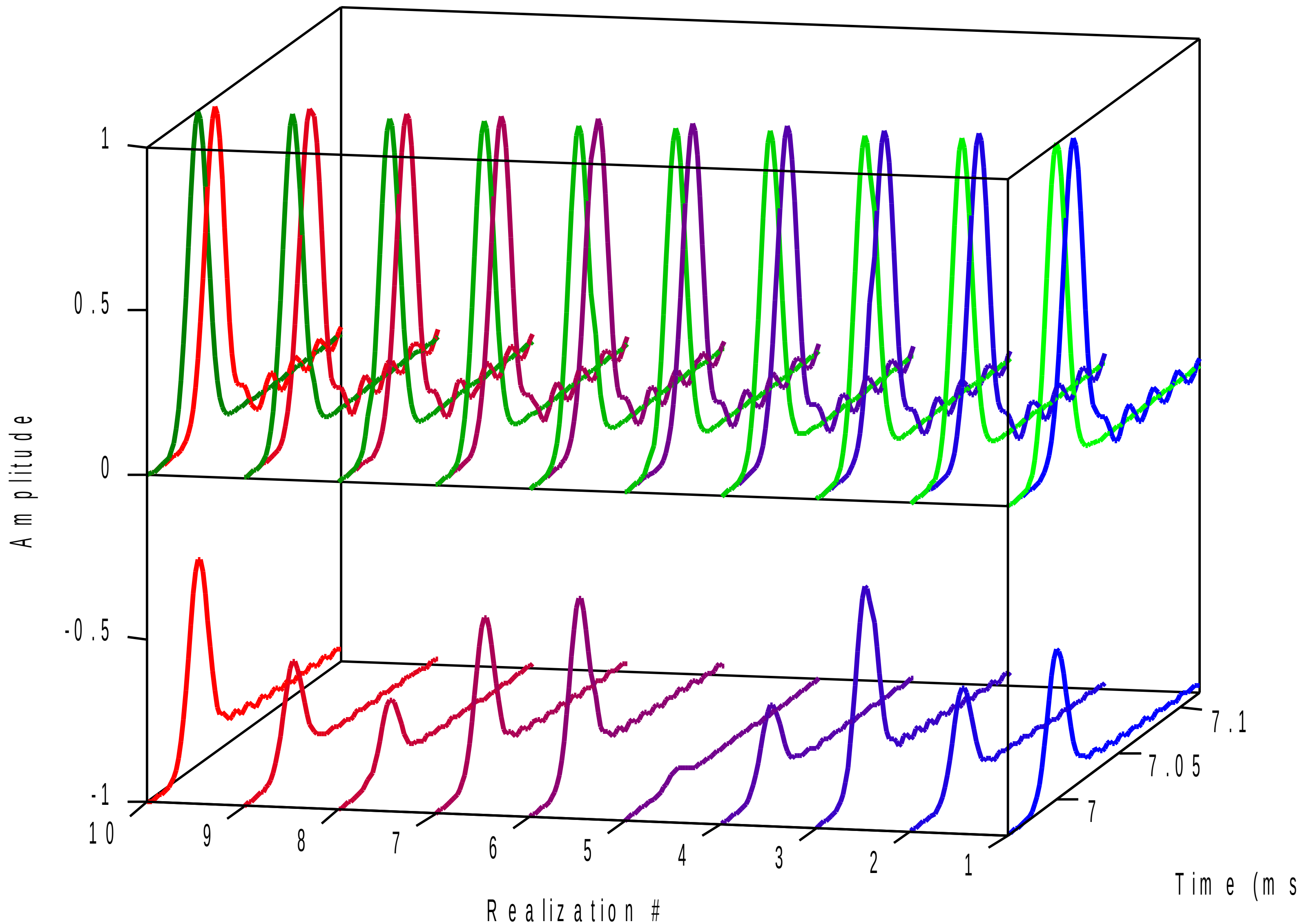
⇒ Non-linear model equation in uncertain quantities

⇒ Uncertainty in time or over realizations???

Stochastic dynamic model 1(2)
1. (Rapid) fluctuations in time?



Stochastic dynamic model 2(2)
2. Fixed for each realization?



Transformed model equation of measurement 1(2)

Continuous time CT

$$\left(\tilde{b}_0 + \tilde{b}_1 s + \tilde{b}_2 s^2 + \tilde{b}_3 s^3 + \ldots\right) Y(s) = \left(\tilde{a}_0 + \tilde{a}_1 s + \tilde{a}_2 s^2 + \tilde{a}_3 s^3 + \ldots\right) X(s)$$

ADC of signal & system – Mapping (impulse-, bilinear, ident....)

$$z = \mathrm{M}(s), \qquad \{a_k, b_k\} = \Lambda\left(\{\tilde{a}_k, \tilde{b}_k\}\right)$$

Discrete time DT

$$\left(b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \ldots\right) Y(z) = \left(a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \ldots\right) X(z)$$

=> Transfer functions (!)

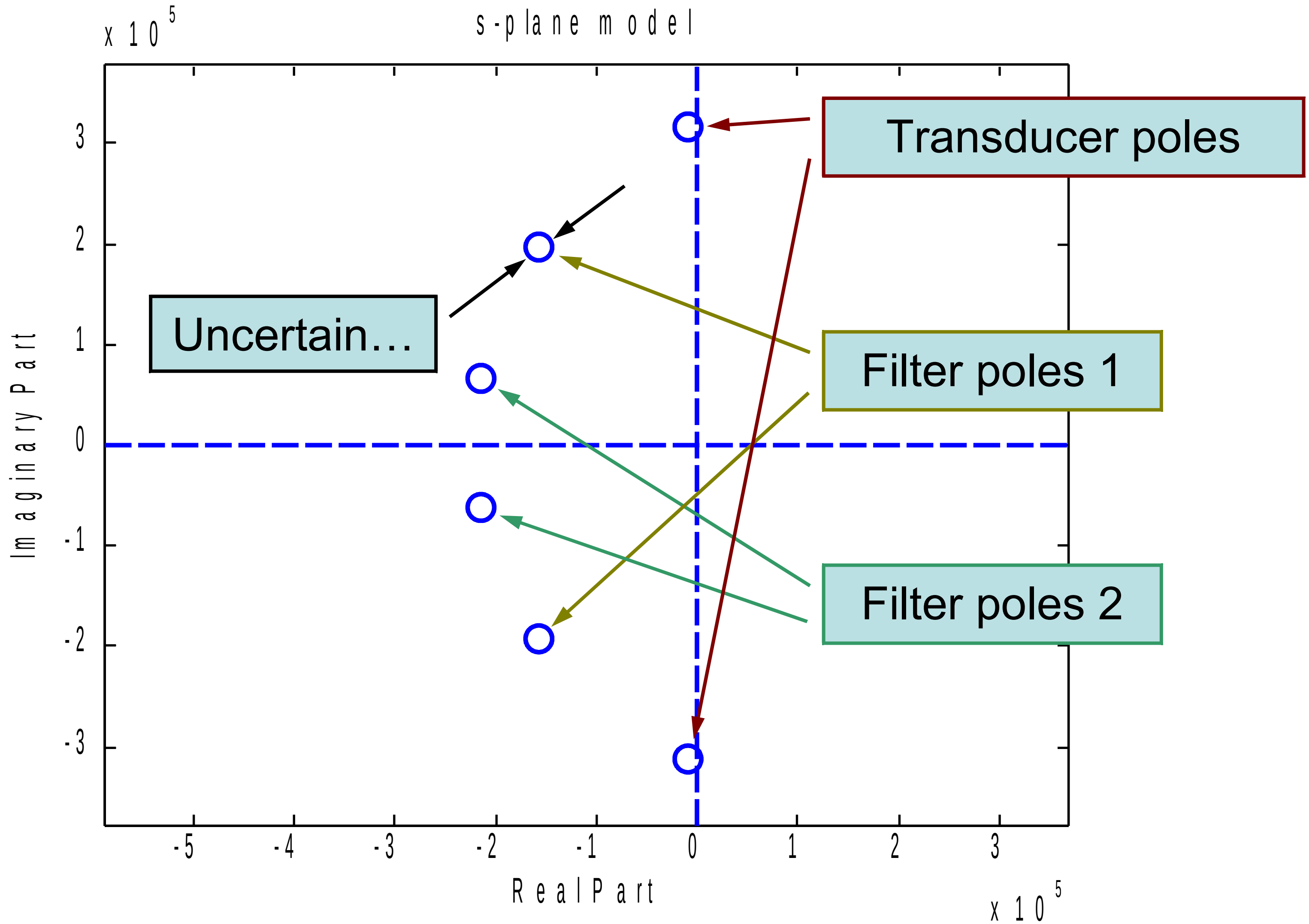
$$CT: \quad H(s) = \frac{X(s)}{Y(s)} = \frac{\tilde{b}_0 + \tilde{b}_1 s + \tilde{b}_2 s^2 + \tilde{b}_3 s^3 + \ldots}{\tilde{a}_0 + \tilde{a}_1 s + \tilde{a}_2 s^2 + \tilde{a}_3 s^3 + \ldots} \equiv \frac{\prod s - \tilde{z}_m}{\prod s - \tilde{p}_l}$$

$$DT: \quad G(z) = \frac{X(z)}{Y(z)} = \frac{b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \ldots}{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots} \equiv \frac{\prod s - z_m}{\prod s - p_l}$$

Zeros

Poles

The dynamic model
– stochastic parameters in the ’s-plane’/’z-plane’ 2(2)



Error propagator (bare) 1(3)

Dynamic model equation $$H(s) = K\frac{\prod s-\tilde{z}_m}{\prod s-\tilde{p}_l},\quad X(s) = H(s)\cdot Y(s)$$

Linearization / factorization

$$\begin{aligned}
\Delta H(s) &= \Delta E(s)\,H(s)\\
\Delta E(s) &= -\sum_m \frac{\Delta\tilde{z}_m}{s-\tilde{z}_m} + \sum_l \frac{\Delta\tilde{p}_l}{s-\tilde{p}_l} \equiv \sum_q \rho_q \frac{|q|}{s-q} \equiv \sum_q \rho_q E_q(s)\\
\rho_q &= \frac{\Delta q}{|q|},\quad q = \{-\tilde{z}_m, \tilde{p}_l\}
\end{aligned}$$

Apply $H^{-1}(s)$ to find error of estimation

– Error propagator acts on **estimated measurand** $\hat{y}(t)$ **:**

$$\Delta\hat{y}(t) = \sum_q \rho_q \cdot e_q(t) * \hat{y}(t)$$

BUT: Complex-valued poles/zeros completely correlated in pairs! (real-valued signals…)

Error propagator – correlated poles/zeros

Include correlation explicitly – combine conjugated pairs

$\mathrm{Im}(q) \geq 0$

Three error subsystems (labelled by IIR orders)

Real-valued par. => '01'

Complex-valued par. => '02', '12'

	$s << \lvert q \rvert$	$s >> \lvert q \rvert$
$E_q^{(01)}(s) = \dfrac{\lvert q \rvert}{s - q}$	~ 1	$\sim \lvert q \rvert / s$
$E_q^{(02)}(s) = \dfrac{\lvert q \rvert^2}{(s - q)(s - \bar{q})}$	~ 1	$\sim \lvert q \rvert^2 / s^2$
$E_q^{(12)}(s) = \dfrac{s \lvert q \rvert}{(s - q)(s - \bar{q})}$	$\sim s / \lvert q \rvert$	$\sim \lvert q \rvert / s$

Error propagator (final) => Error model 3(3)

Transfer function $E_q^{(mn)}(s)$ ⇔ Impulse response $e_q^{(mn)}(t)$

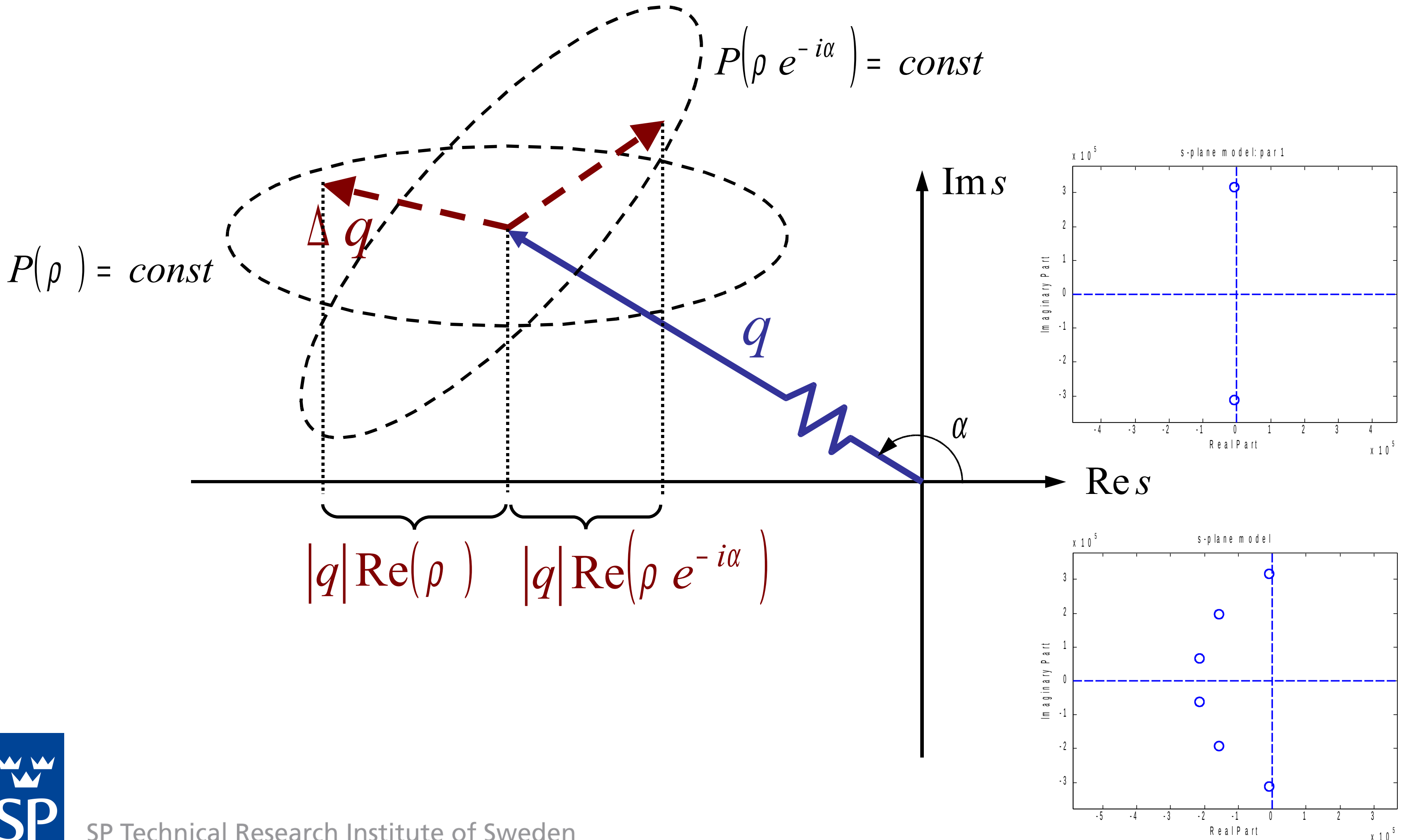
$$\varsigma_q^{(mn)}(\hat{y}) = e_q^{(mn)}(t) * \hat{y}(t)$$

$$\rho_q = \frac{\Delta q}{|q|}$$

$$q = \{z_m, p_m\} = |q| e^{i\alpha}$$

$$\Delta \hat{y}(t) = \begin{cases} \displaystyle\sum_{\operatorname{Im} q = 0} \rho_q \varsigma_q^{(01)}(\hat{y}) + \\ \displaystyle 2 \sum_{\operatorname{Im} q > 0} \left[\begin{array}{l} \operatorname{Re}(\rho_q) \varsigma_q^{(12)}(\hat{y}) \\ - \operatorname{Re}(\rho_q e^{-i\alpha}) \varsigma_q^{(02)}(\hat{y}) \end{array} \right] \end{cases}$$

Statistical analysis
– Dynamic model errors $\rho = \Delta q / q$ 1(2)



”The law of propagation of uncertainty…” (GUM) 2(2)

Assume

- All $\left\{\rho_q = \Delta q / |q|\right\}$ uncorrelated
- All probability density distributions $P\left(\rho_q\right)$ uniform in angle

- Uncertainties combined **similarly** static/dynamic case

$$u_c^2\left(y,t\right) = \sum_{\mathrm{Im}\, q \geq 0} u_q^2 c_q^2 \rightarrow \sum_{\mathrm{Im}\, q \geq 0} u_q^2 c_q^2\left(\hat{y}\left(t\right)\right)$$

- Real-valued poles/zeros:

$$c_q\left(\hat{y}\right) = \varsigma_q^{(01)}\left(\hat{y}\right)$$

Filtering:

$$\varsigma_q^{(mn)}\left(\hat{y}\right) = e_q^{(mn)}\left(t\right) * \hat{y}\left(t\right)$$

- Complex-valued poles/zeros ’paired’:

$$c_q\left(\hat{y}\right) = 2 \cdot \left| \varsigma_q^{(02)}\left(\hat{y}\right) - \exp\left(i\alpha\right) \varsigma_q^{(12)}\left(\hat{y}\right) \right|$$

Realization of propagator of uncertainty – Digital filter

1. Map all continuous time poles and zeros* $q = \exp\left(\tilde{q}\, T_S\right)$

3. Adjust amplification

5. Read off input $\{b_m\}$ output $\{a_l\}$ filter coefficients $E_q(z) = \dfrac{\sum_m b_m z^{-m}}{\sum_l a_l z^{-l}}$

Continuous time s-plane

$$E_q^{(01)}(s) = \frac{|q|}{s - q}$$

$$E_q^{(02)}(s) = \frac{|q|^2}{(s - q)(s - \bar{q})}$$

$$E_q^{(12)}(s) = \frac{s|q|}{(s - q)(s - \bar{q})}$$

Discrete time z-plane

$$E_q^{(01)}(z) = \frac{|q| T_S}{z - \exp(q T_S)}$$

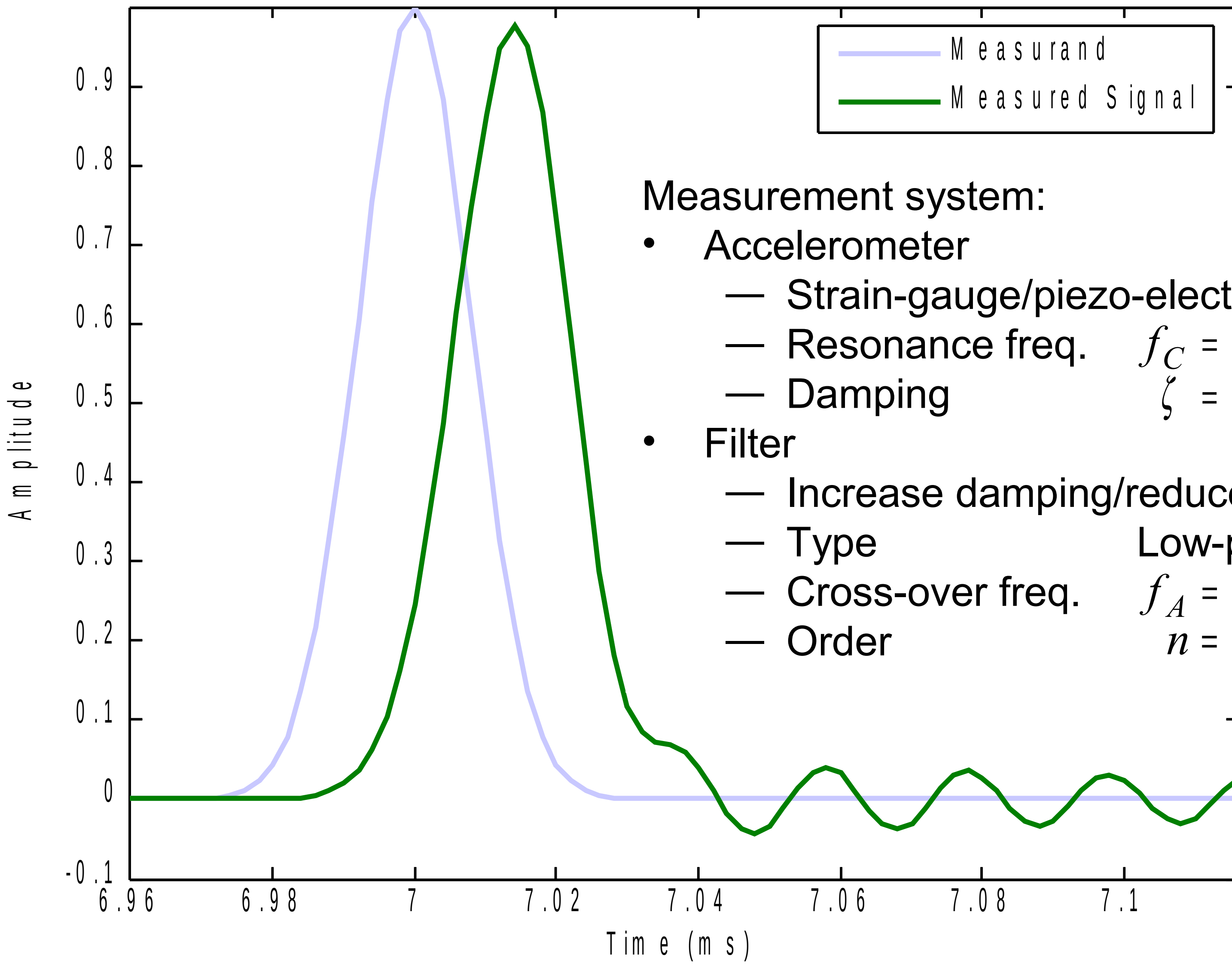
$$E_q^{(02)}(z) = \frac{|q|^2 T_S^2}{[z - \exp(q T_S)][z - \exp(\bar{q} T_S)]}$$

$$E_q^{(12)}(z) = \frac{(z - 1)|q| T_S}{[z - \exp(q T_S)][z - \exp(\bar{q} T_S)]}$$

*J. P. Hessling 2008 A novel method of dynamic correction *Meas.Sci.Technol.* **19** 075101

Example:
Transducer system…

The dynamic measurement



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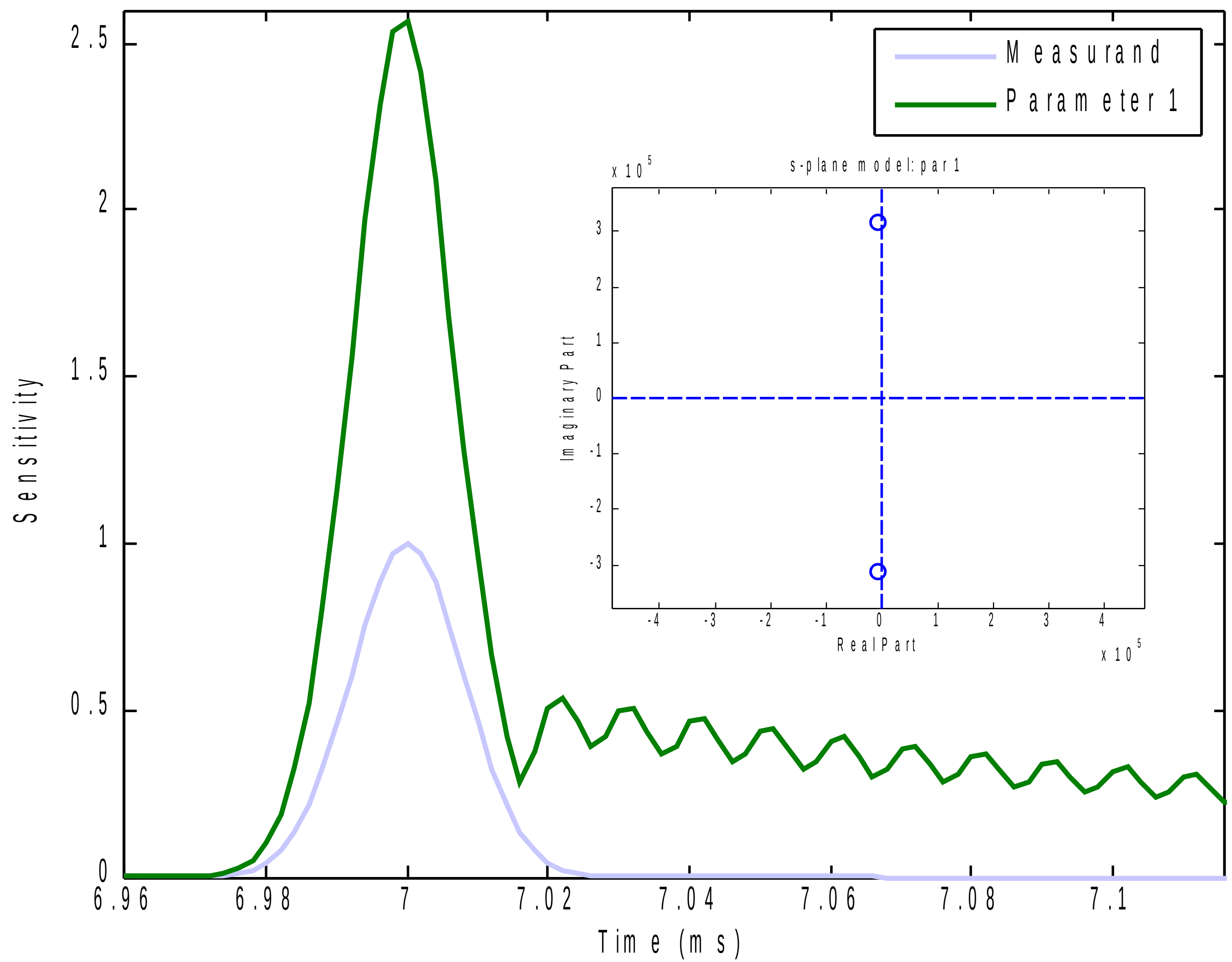
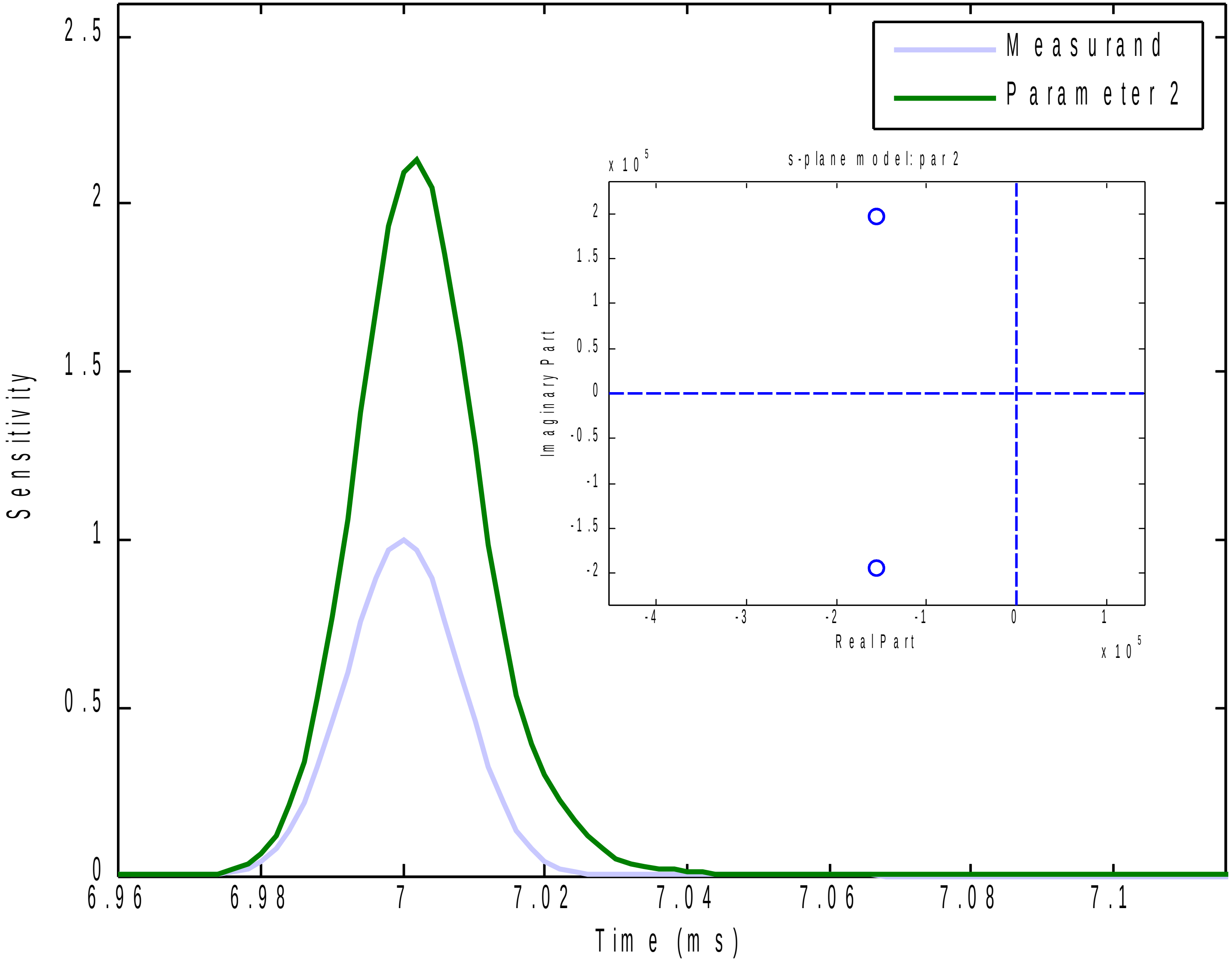
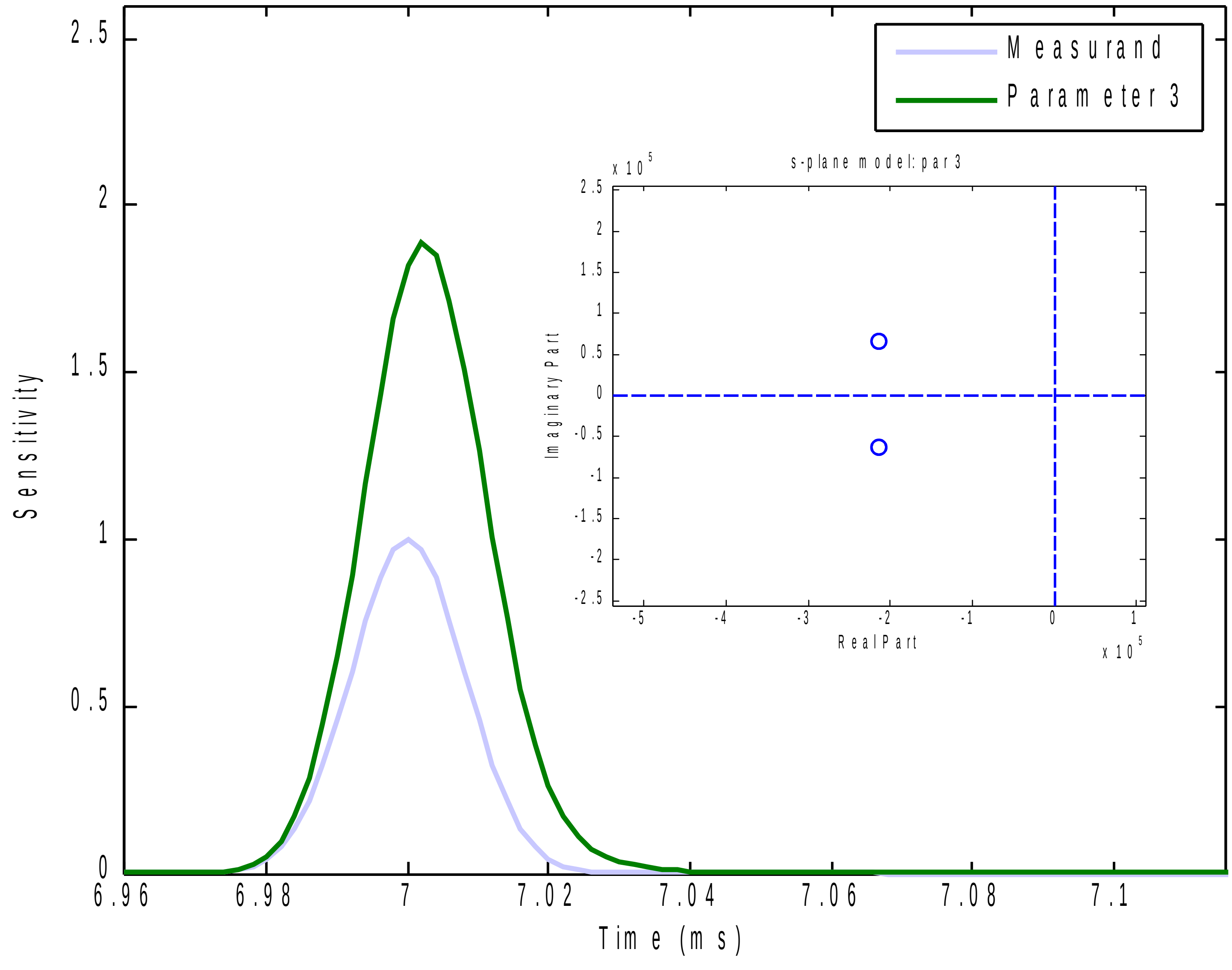
Dynamic Model Uncertainty
Alt. 1: Sensitivity vs. time, contr. 1/3 – Transducer



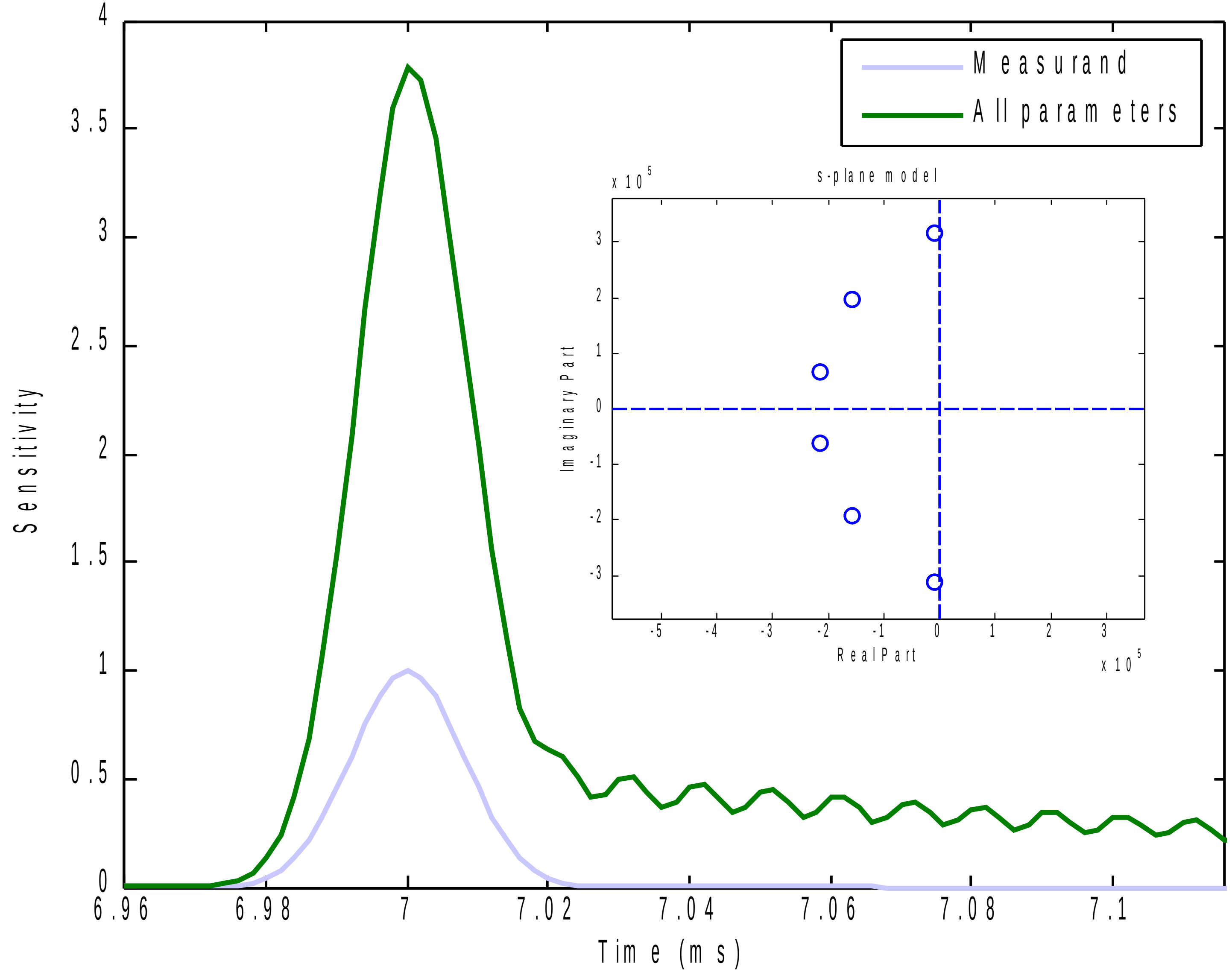
Dynamic Model Uncertainty:
Alt. 1: Sensitivity vs. time, contr. 2/3 – Filter poles 1

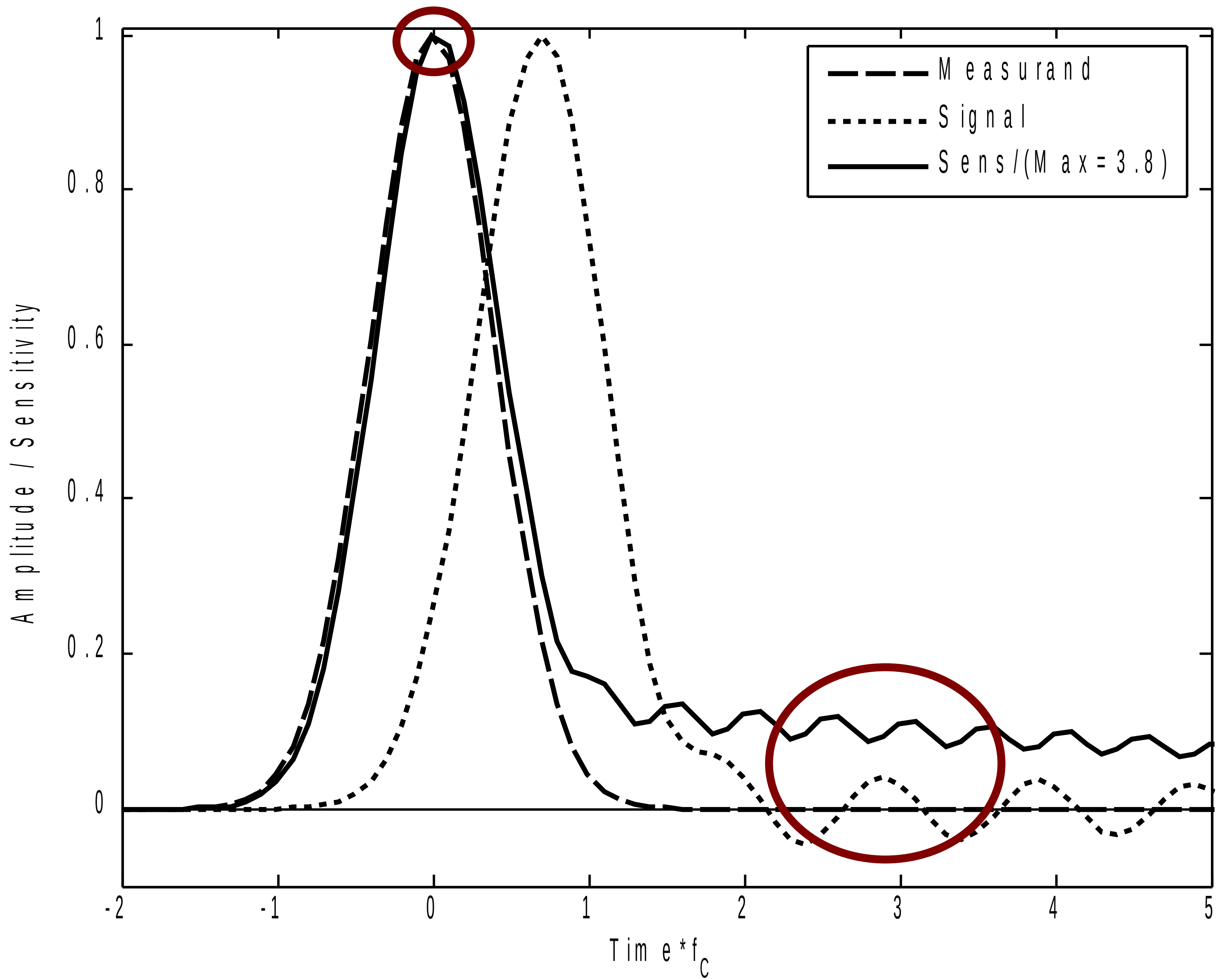


Dynamic Model Uncertainty
Alt. 1: Sensitivity vs. time, contr. 3/3 – Filter poles 2



Dynamic Model Uncertainty
Alt. 1: Total sensitivity vs. time (equal par. unc.)





Measurand
Signal
Sens/(Max=3.8)
Amplitude / Sensitivity
Time*f_C
1
0.8
0.6
0.4
0.2
0
-2
-1
0
1
2
3
4
5

Summary

Dynamic model uncertainty…

- Contribution to **dynamic uncertainty:**
  - Uncertain dynamic model poles/zeros
- Procedure
  - Linearization s-plane model
  - Statistical analysis, ensemble averages
  - Propagator of uncertainty: Parameter unc.=> Measurement unc.
- **Similarity** static/stationary unc.
  - **Linearization – Sensitivity**
  - Quadratic summation
  - Time-invariant measurement system
- **Difference** static/stationary unc.
  - Time-dependent measurand => Time-dependent **sensitivity**
  - Linearization in s-plane, not time
  - Physical correlation: complex-conjugated pole/zero pairs
- Excluded
  - System Identification
  - Estimation measurand / dynamic error

Thanks for your attention!

The large picture…

What is it all about?...

Holistic perspective – Dynamic metrology

- **Characterization** / 'Dynamic calibration'
  => ***Complete** dynamic description*
- **System identification**
  => *Dynamic model*
- **Error bound estimation**
  => *Systematic/repeatable error*
- **Uncertainty evaluation**
  => *Uncertain deviation*

(Error bound estimation and Uncertainty evaluation) **=> Combined uncertainty**

- **System optimization**
  => *Reduced error*
- **Dynamic correction / estimation**
  => *Eliminate(?) error and time delay*

Studied here:

- **Uncertainty evaluation**
  - **Time-dependence of measurand**
    - **Dynamic model parameters**
      (measurement noise, model errors, implementation uncertainty etc. excluded)

Sensitivity coefficients / functions / signals

GUM:

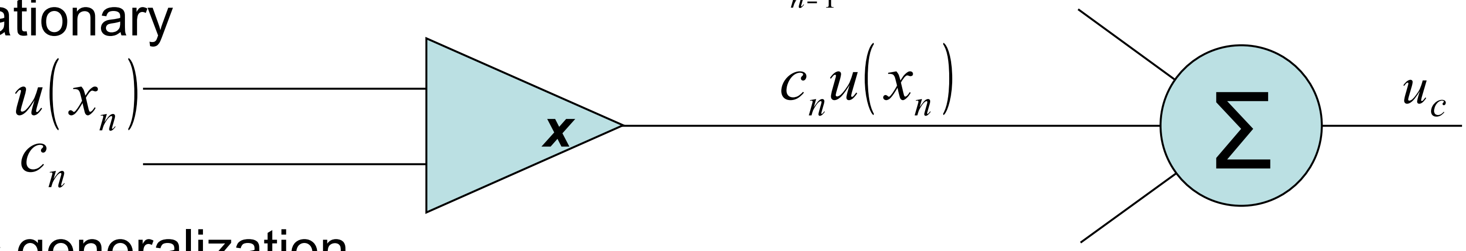
- Model (algebraic?!) of measurement
- Sensitivity coefficients
- Combined uncertainty

$$Y = f\left(X_1, X_2, \ldots, X_N\right)$$

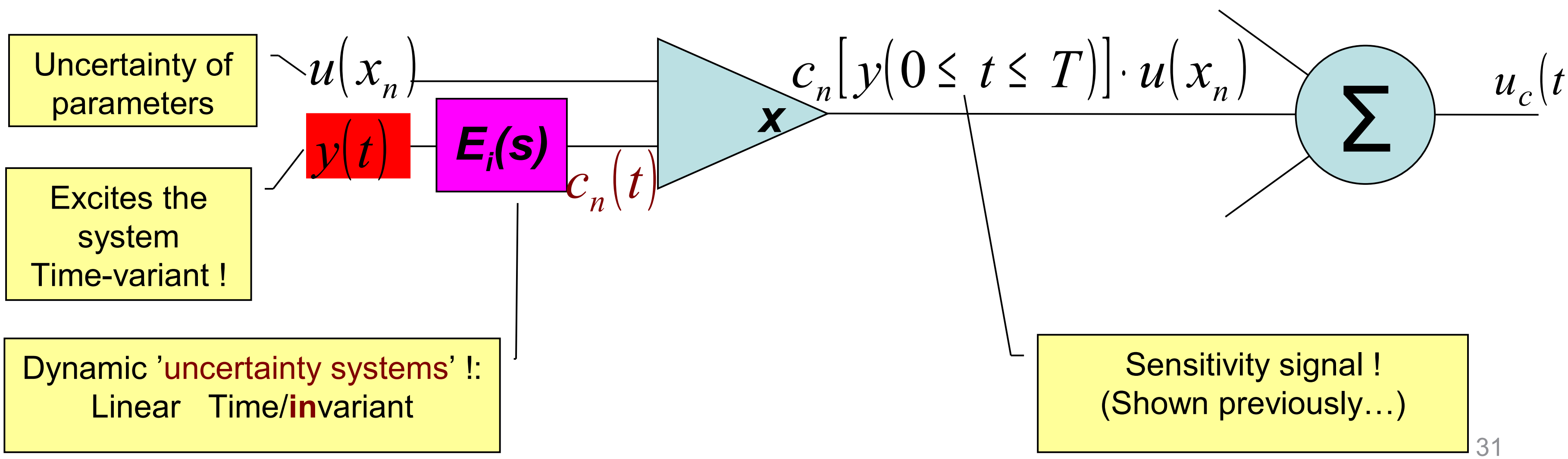
$$c_n = \frac{\partial f}{\partial x_n}$$

$$u_c^2 = \sum_{n=1}^{N} c_n^2 u^2\left(x_n\right)$$

Static/stationary



Dynamic generalization



"The law of propagation of uncertainty…" (GUM) 3(3)

Assume

- All $\{\rho_k = \Delta q_k / |q_k|\}$ uncorrelated
- All probability density distributions $P(\rho_k)$ uniform in angle

Time-dependent uncertainty:

$$u_c^2(y,t) = \sum_{\mathrm{Im}\,q \geq 0} u_q^2 c^2(t,q)$$

$$c(t,q) = \begin{cases} \varsigma_{01}(t,q) & , \quad \mathrm{Im}(q) = 0 \\ 2 \cdot \left|\varsigma_{02}(t,q) - \exp(i\alpha)\varsigma_{12}(t,q)\right| & , \quad \mathrm{Im}(q) > 0 \end{cases}$$

$$\langle \rho_k \rho_l \rangle = \delta_{kl} \cdot u_{q_k}^2, \quad \langle \rho_k \rangle = 0$$

Propagator of uncertainty – sensitivity signals *c(t,q)*

$$u_c^2(y) = \sum_{\operatorname{Im} q \geq 0} u_q^2 c^2(t,q)$$

$$c(t,q) = \begin{cases} \varsigma_{01}(t,q) & , \quad \operatorname{Im}(q) = 0 \\ 2 \cdot \left[\begin{array}{l} \varsigma_{02}^2(t,q) + \varsigma_{12}^2(t,q) \\ - 2\cos(\alpha)\varsigma_{02}(t,q)\varsigma_{12}(t,q) \end{array} \right]^{1/2} & , \quad \operatorname{Im}(q) > 0 \end{cases}$$

$$\varsigma_{mn}(t,q) = e_{mn}(t,q) * y(t)$$

Summation rule

- Uncertainties combined **similarly** static/dynamic case

$$u_c^2\left(y,t\right) = \sum_{\operatorname{Im} q \geq 0} u_q^2 c^2(q) \rightarrow \sum_{\operatorname{Im} q \geq 0} u_q^2 c^2(t,q)$$

- Real-valued poles/zeros:

$$c(t,q) = \varsigma_{01}(t,q)$$

- Complex-valued poles/zeros ’paired’:

$$c(t,q) = 2 \cdot \left| \varsigma_{02}(t,q) - \exp\left(i\alpha\right) \varsigma_{12}(t,q) \right|$$

Statistical averages and ergodicity? 2(3)

- Static/stationary measurements:
  - Time average
  - Ensemble average

Equal – **ergodic** systems

- Dynamic measurements:

$$\langle \rho_l \rho_m \rangle = \left\langle \frac{\Delta q_l}{q_l} \frac{\Delta q_m}{q_m} \right\rangle$$

  - Time average???
    - Time-***dependent*** measurand / ***uncertainty (!)***
  - Ensemble average
    - Repeat measurement **same** measurand???
      - Monte Carlo simulation
      - Statistical evaluation of model

Time-**variant** measurement – **non-ergodic** system!