Current Issues in the Evaluation of Ionising Radiation Measurements

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Simplest Case: Net Count Rate Measurement

Probability distributions with and without sample

Gross measurement: $$f(n_g \mid \rho_g, t_g) = \frac{(\rho_g t_g)^{n_g} e^{-\rho_g t_g}}{n_g!}$$

Background measurement: $$f(n_0 \mid \rho_0, t_0) = \frac{(\rho_0 t_0)^{n_0} e^{-\rho_0 t_0}}{n_0!}$$

t ... Counting time
ρ ... True value of the count rate
n ... Number of pulses counted

Mathematical model of the measurement

True value $\tilde{y}$ of the measurand: $\tilde{y} = \rho_g - \rho_0$

Decision Threshold: Conventional vs. Bayesian

Conventional statistics [Currie 1968]

- Count rate estimate: $y = n_g/t_g - n_0/t_0$
- Considering the distribution of y for $\rho_g = \rho_0$
- Gaussian approximation
- Estimation of ρ_0 by n_0/t_0

$$y^* = k_{1-\alpha} \sqrt{\frac{n_0}{t_0}\left(\frac{1}{t_g} + \frac{1}{t_0}\right)}$$

Bayesian statistics [Weise 1998]

- Posterior distribution of ρ_0 for n_0 measured
- Uniform prior
- Convolution with distribution of n_g for $\rho_g = \rho_0$
- Integrating out ρ_0 and approximating
- Count rate estimate: $y = n_g/t_g - (n_0 + 1)/t_0$

$$y^* = k_{1-\alpha} \sqrt{\frac{n_0 + 1}{t_0}\left(\frac{1}{t_g} + \frac{1}{t_0}\right)}$$

$k_{1-\alpha}$ ... $(1 - \alpha)$-Quantile of the normal distribution

More Complex Cases: Activity Measurements

Application of a calibration factor ε:

$$\tilde{y} = \frac{\rho_g - \rho_0}{\varepsilon}$$

$\tilde{y}$ ... Activity
ε ... Detection efficiency (= ratio of net count rate and activity)

Wipe test:

$$\tilde{y} = \frac{\rho_g - \rho_0}{\varepsilon\, \eta\, A}$$

$\tilde{y}$ ... Specific activity (per area)
ε ... Detection efficiency (= ratio of count rate and activity collected)
η ... Wiping efficiency (= portion of the activity taken off)
A ... Area wiped

➔ Other uncertainty contributions than just counting statistical uncertainty

➔ Measurement model may comprise quantities with Type A and such with Type B uncertainties

Leading up to ISO/DIS 11929:2008

Concept: [Weise 1998], based on [Weise & Wöger 1992]

Objectives of the new approach:

- Taking all uncertainty contributions into account, not only counting statistics
- Employing Bayesian statistics (motivated by the need to combine Type A and Type B uncertainties)
- Decision threshold and detection limit shall not differ significantly from conventional statistics (enabling to maintain existing practices)

Applied to the parts of ISO 11929 issued from 2005 onwards

To be used for replacement of the current ISO 11929 series

General framework of ISO/DIS 11929:2008

Translation of its Annex F into common terminology

- Non-radiating sample corresponds to the smallest true value $\tilde{y} = 0$
  $\Rightarrow$ Parameter space $\Pi = [0,\infty)$
- Observed value y may be negative ("Primary measurement result")
  $\Rightarrow$ Sample space $\Sigma = (-\infty,\infty)$
- Conditional probability $f(y|\tilde{y})$ of the estimate y given the true value $\tilde{y}$
  $\Rightarrow$ Function $f(y|\tilde{y})$ of test statistic y for fixed $\tilde{y}$ called sampling distribution
  $\Rightarrow$ Function $f(y|\tilde{y})$ of $\tilde{y}$ for constant y is the likelihood function
- $f_0(\tilde{y}|y)$ symbolizes the likelihood in ISO/DIS 11929:2008, not e.g. $L(\tilde{y};y)$
  $\Rightarrow$ $f_0(\tilde{y}|y) = f(y|\tilde{y})$ for $\tilde{y} \in \Pi$, $y \in \Sigma$
- Conditional probability $f(\tilde{y}|y)$ of the true value $\tilde{y}$ given the estimate y
  $\Rightarrow$ Function $f(\tilde{y}|y)$ of $\tilde{y}$ for a particular y is the posterior distribution

Extension of the Parameter Space

Bayes' theorem in its usual form

$$f(\tilde{y}\,|\,y) = \frac{f_0(\tilde{y}\,|\,y)\,f(\tilde{y})}{\int_\Pi f_0(\tilde{y}'\,|\,y)\,f(\tilde{y}')\,d\tilde{y}'},\ \tilde{y}\in\Pi,\ y\in\Sigma$$

Uniform ordinary prior $f(\tilde{y})$ advocated

Bayes' theorem for extended parameter space Ξ = (-∞,∞)

$$f(\tilde{y}\,|\,y) = \frac{f_0(\tilde{y}\,|\,y)\,f(\tilde{y})}{\int_\Xi f_0(\tilde{y}'\,|\,y)\,f(\tilde{y}')\,d\tilde{y}'},\ \tilde{y}\in\Xi,\ y\in\Sigma$$

- $f(\tilde{y})$ extended to a Heaviside step function, zero for $\tilde{y} < 0$, unity for $\tilde{y} \geq 0$
- Prior not strictly positive as in usual Bayesian statistics [Bernardo 2003]
- Likelihood function formally extended to $\tilde{y} < 0$, but irrelevant there
- What does probability $f(y|\tilde{y})$ of y for an impossible true value $\tilde{y} < 0$ mean?

Applying of the Principle of Maximum Entropy

Extended likelihood function of maximum entropy

- Extended parameter space Ξ = Range of integration
- Primary measurement result *y* = Constraint for expectation
- Its squared combined standard uncertainty $u^2(y)$ = Constraint for variance

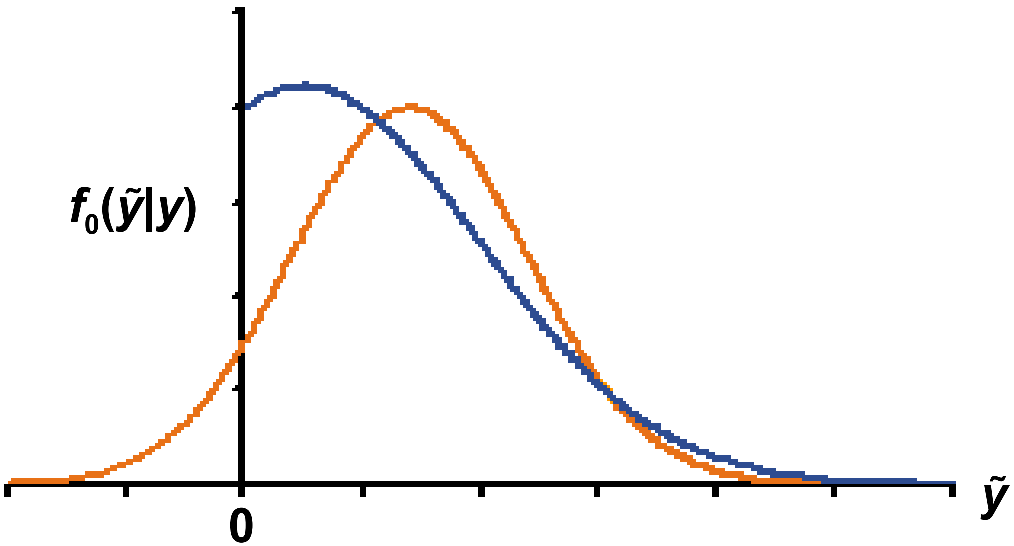
➔ Gaussian distribution for the extended likelihood function:

$$f_0(\tilde{y} \mid y) \propto \exp\left(-\frac{(\tilde{y} - y)^2}{2u^2(y)} \right), \tilde{y} \in \Xi, y \in \Sigma$$

- $u^2(y)$ calculated by combining the (Type A and Type B) standard uncertainties of the estimates of the input quantities [GUM 1995]
- Motivation for this approach: Using such a $u^2(y)$ apparently considered theoretically sound for the likelihood but not for the sampling distribution

Impact of Extending the Parameter Space

PME distribution for ordinary and extended parameter space



- PME for ordinary parameter space:
  Broader, left-shifted and truncated Gaussian distribution [Jaynes 1968]
- Extending the parameter space has decisive effect on maximum-entropy distribution

Sampling Distribution for Combined Uncertainty

Ensuing extended posterior distribution

$$f(\tilde{y} \mid y) \propto \exp\left(-\frac{(\tilde{y} - y)^2}{2u^2(y)} \right) \cdot f(\tilde{y}),\ \tilde{y} \in \Xi,\ y \in \Sigma$$

Bayes' theorem for extended parameter space

Expression using the sampling distribution instead of the likelihood function:

$$f(\tilde{y} \mid y) \propto f(y \mid \tilde{y}) f(\tilde{y}),\ \tilde{y} \in \Xi,\ y \in \Sigma$$

Sampling distribution obtained by comparison

$f(y|\tilde{y})$ only derived for $\tilde{y} \geq 0$, undetermined for $\tilde{y} < 0$ since $f(\tilde{y}) \equiv 0$ for $\tilde{y} < 0$:

$$f(y \mid \tilde{y}) \propto \exp\left(-\frac{(\tilde{y} - y)^2}{2u^2(y)} \right),\ \tilde{y} \in \Pi,\ y \in \Sigma$$

Function of *y* for given $\tilde{y}$ not Gaussian

Final Form of the Sampling Distribution

Alternative derivation without invoking Bayes' theorem

- Passing from sampling distribution to the likelihood due to $f_0(\tilde{y}|y) = f(y|\tilde{y})$
- Extension of parameter space for likelihood function
- Calculation of PME likelihood function and return to sampling distribution

➔ Using Bayes' theorem should probably impart a Bayesian character to the likelihood to justify combining Type A and Type B uncertainties.

Approximate formula of the sampling distribution

- Variance $u^2(y)$ of the function $f_0(\tilde{y}|y)$ of $\tilde{y}$ for primary measurement result y approximated by variance $\tilde{u}^2(\tilde{y})$ of function $f(y|\tilde{y})$ of y for true value $\tilde{y}$
- Justified by assuming y to be close to $\tilde{y}$ and weak dependence of u^2 on y

$$f(y\,|\,\tilde{y}) \propto \exp\left(-\frac{(\tilde{y}-y)^2}{2\tilde{u}^2(\tilde{y})}\right), \tilde{y} \in \Pi\,, y \in \Sigma$$

Gaussian distribution of y for given $\tilde{y}$

Decision Threshold and Detection Limit

Usual general equations for Gaussian sampling distribution

Decision threshold y^*: $y^* = k_{1-\alpha}\,\tilde{u}(0)$

Detection limit $y^\#$: $y^\# = y^* + k_{1-\beta}\,\tilde{u}(y^\#)$

$k_{1-\alpha}$, $k_{1-\beta}$ ... Quantiles of the normal distribution

Case of net count rate measurement

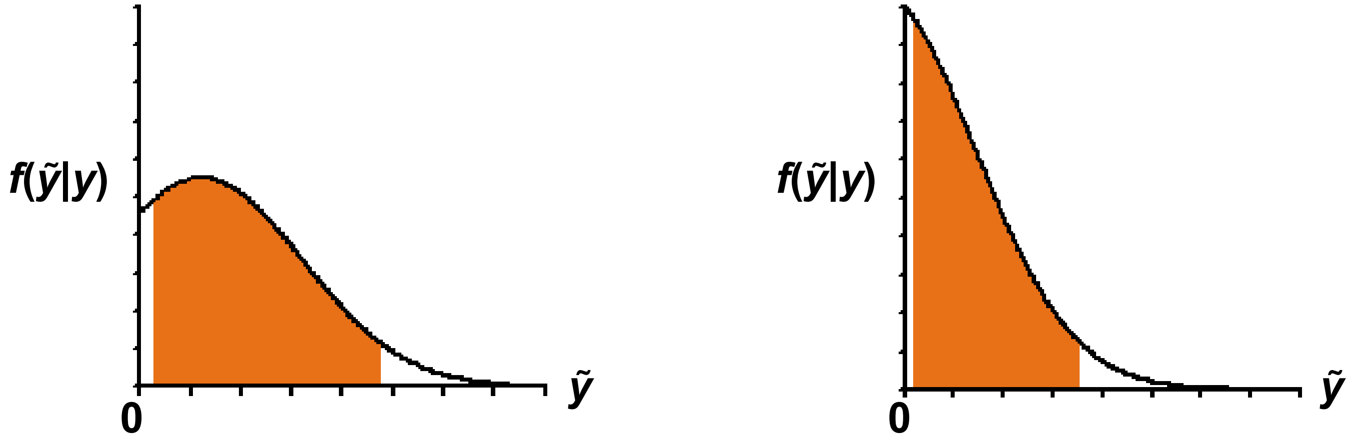
$$y^* = k_{1-\alpha}\sqrt{\frac{n_0}{t_0}\left(\frac{1}{t_g}+\frac{1}{t_0}\right)}$$

- Identical to the conventional statistics approach, not to the Bayesian!
- For small value ($\rho_0 t_0$) probability of type I error deviates more strongly from α than for the Bayesian equation [Strom & MacLellan 2001]

Coverage Interval for the Measurand

Probabilistically symmetric coverage interval stipulated

Never includes zero and sometimes not even the posterior mode:



- Seems to indicate that regardless of coverage probability chosen complete absence of radiation could be excluded.
- Such an interpretation might cause unwarranted irritation.
- Avoided by interval of shortest length, mathematically simple for truncated Gaussian distribution [Little 1982, Roe & Woodroofe 2001]

Conclusion

Review of the ISO/DIS 11929:2008 approach

- Favourable intention to consider all uncertainty contributions for calculating the sampling distribution
- Questionable concept of extending the parameter space and the likelihood function into unphysical region
- Arbitrary choice of setting up just the extended likelihood function by the primary measurement result and the combined standard uncertainty
- Approach not theoretically better substantiated than sampling distribution calculation by the usual combination of Type A and Type B uncertainties

Some References

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