Allan variance and the Uncertainty of Autocorrelated Measurements

Nien Fan Zhang

National Institute of Standards and Technology

Gaithersburg, MD 20899, USA

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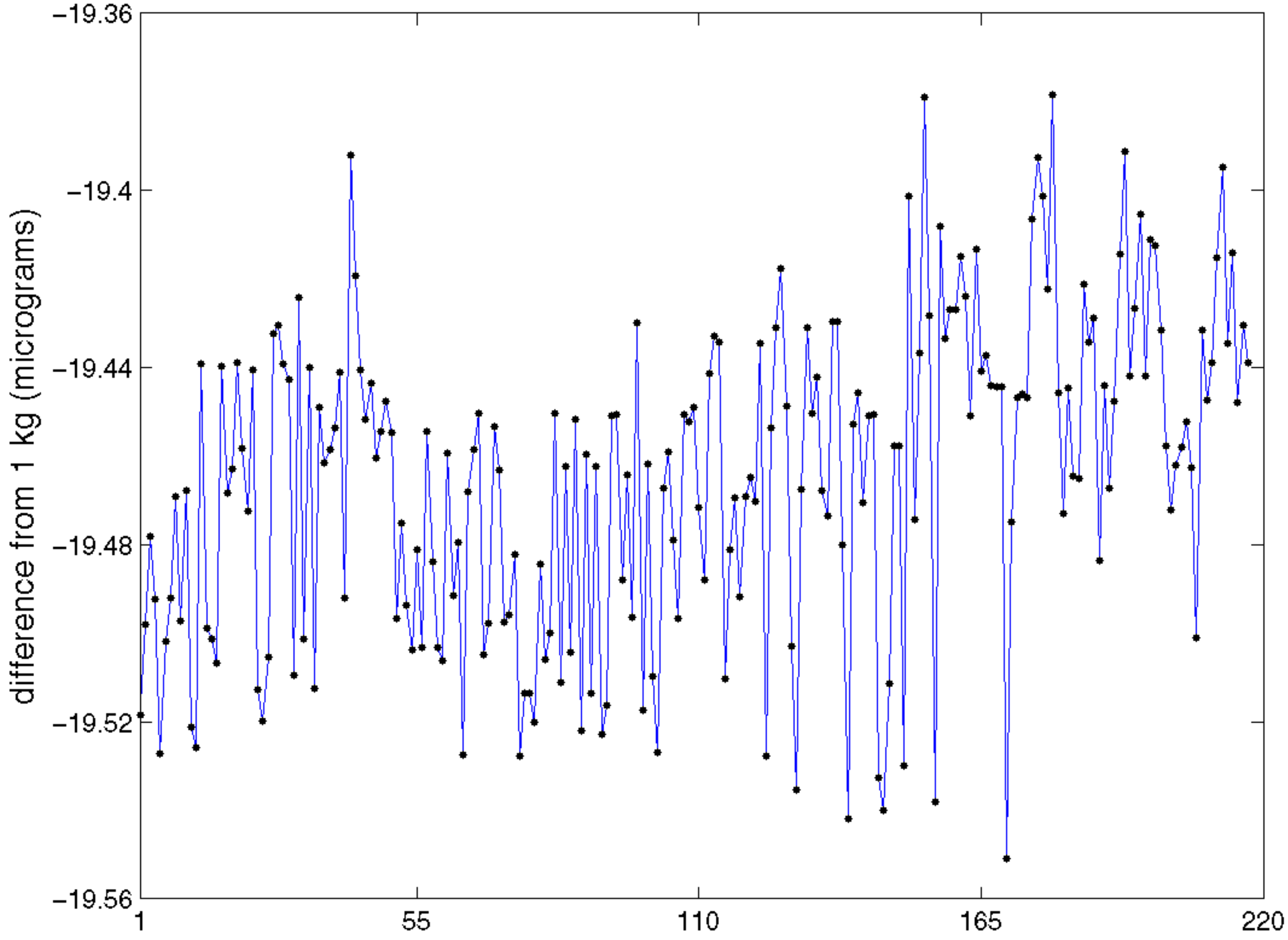
1. Introduction

In metrology, for given repeated measurements, $\{X(1),...,X(n)\}$ of size of n

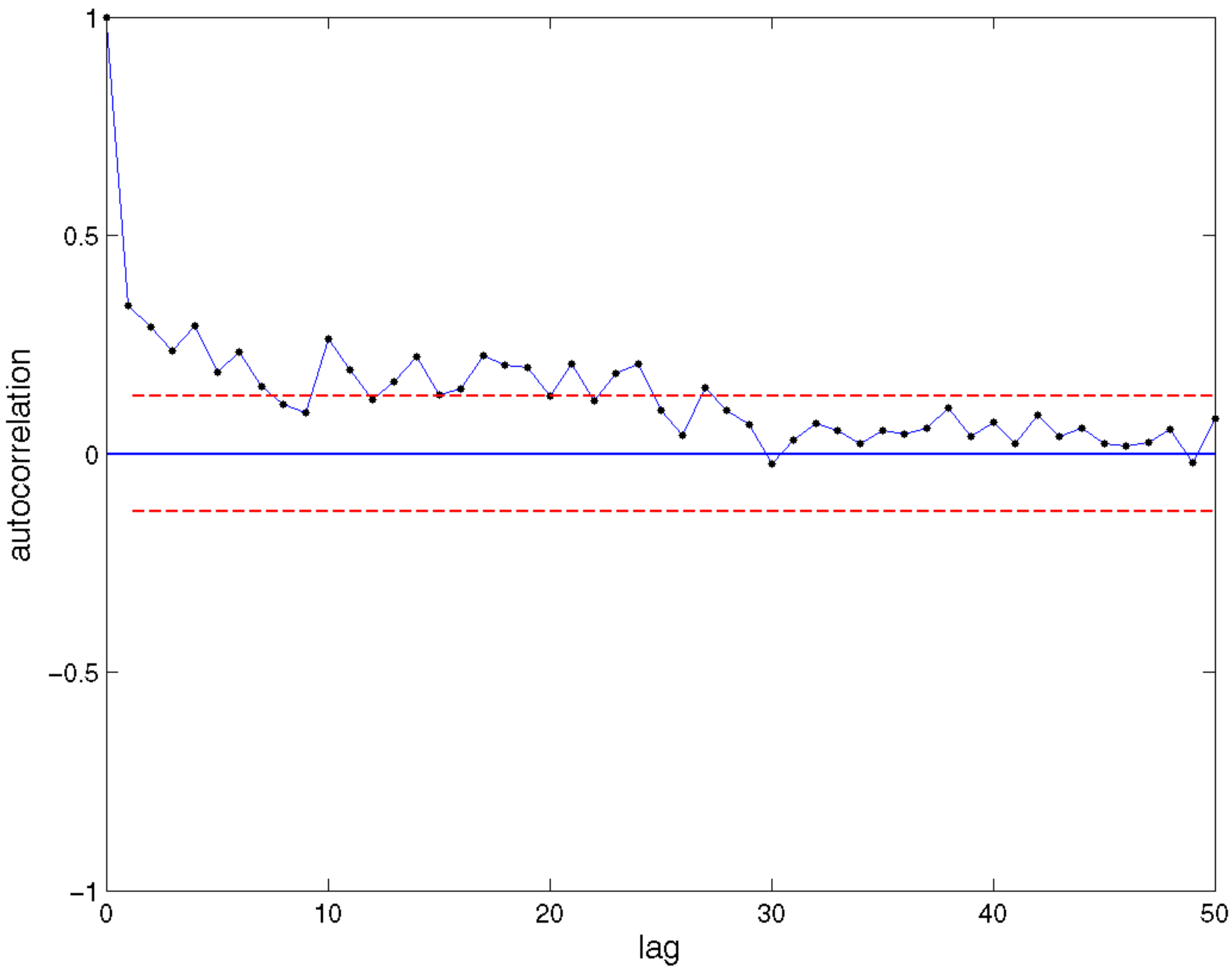
$$u_{\bar{X}} = \frac{S_X}{\sqrt{n}}$$

When n increases, the uncertainty of the sample mean reduces. Here we assume that $\{X(1),...,X(n)\}$ have same mean and variance and are uncorrelated. However the condition of uncorrelatedness does not always hold. In many cases measurements are autocorrelated or self-correlated. Here is an example.

High precision weight measurements with differences from the 1kg check standards.



This shows the ACF plot of the weight measurements with a 95% confidence band for WN.



The ACF plot is a plot for the sample autocorrelations, which measures the autocorrelation of the data.

The confidence band is centered at zero and with limits of $\pm 1.96/\sqrt{n}$ with $n = 217$ The confidence band is for the autocorrelations of an uncorrelated sequence or white noise. Namely, if the data are statistically independent, then its sample autocorrelation has a mean of zero and an approximate standard deviation of $1/\sqrt{n}$

Obviously, the weight measurements are autocorrelated and thus the uncertainty given in the above may not be appropriate to use.

GUM (1995) pointed out in 4.2.7: “If the random variations in the observations of an input quantity are correlated, for example in time, the mean and experimental standard deviation of the mean as given in 4.21. and 4.2.3 may be inappropriate estimators of the desired statistics.”

Thus, appropriate approaches are needed to calculate the corresponding uncertainties. For autocorrelated processes, an important class is the one of stationary processes. In Zhang (2006), a practical approach to compute the uncertainty of the average of the measurements from a stationary process was proposed. Witt (2007) discussed the use of autocorrelation function to characterize time series of voltage measurements.

2. Stationary processes

A discrete time series $\{X(t), t = 1, 2, \ldots\}$ is (weakly) stationary if

(1) $E[X(t)] = \mu$

(2) $Var[X(t)] = \sigma_X^2 < \infty$

(3) $Cov[X(t), X(t+\tau)] = R(\tau)$

$R(\tau)$ is the autocovariance of $\{X_t\}$ at lag of τ. $R(0) = \sigma_X^2$

The autocorrelation is defined as $\rho(\tau) = R(\tau)/R(0)$

When the measurements are from a stationary process, they have the same mean and variance for all t.

Examples of stationary processes

(2) White noise ---- mean=0 and $\rho(\tau) = 0$ for all $\tau \neq 0$

(3) First order autoregressive process (AR(1))

$$(X(t) - \mu) = \phi(X(t-1) - \mu) + a(t)$$

This can be written as

$$(1 - \phi_1 B)[X(t) - \mu] = a(t)$$

where B is the back shift operator, i.e., $B[X(t)] = X(t-1)$

When $k \geq 0$ $\rho(k) = \phi^k$

When $|\phi| < 1$ the process is stationary.

Uncertainty of a sample mean for stationary measurements

For the sample mean $\bar{X} = \dfrac{X(1) + \ldots + X(n)}{n}$ when $\{X(t)\}$ is stationary

$$Var[\bar{X}] = \left[1 + \frac{2\sum_{i=1}^{n-1} (n-i)\rho(i)]}{n} \right] \frac{\sigma_X^2}{n}$$

where σ_X^2 is the variance of $\{X(t)\}$. When the measurements are uncorrelated,

$$Var[\bar{X}] = \frac{\sigma_X^2}{n}$$

When $\{X(t)\}$ is an AR(1) process

$$Var[\bar{X}] = \frac{n - 2\phi - n\phi^2 + 2\phi^{n+1}}{n^2(1-\phi)^2}\sigma_X^2 : \frac{1}{n}$$

The uncertainty of the sample mean is given by

$$u_{\bar{X}}^2 = \left[1 + \frac{2\sum_{i=1}^{n-1} (n - i)\hat{\rho}(i)]}{n} \right] \frac{S_X^2}{n}$$

where $$S_X^2 = \frac{1}{n-1}\sum_{k=1}^{n} (X(k) - \bar{X})^2$$

$$\hat{\rho}(i) = \frac{\sum_{k=1}^{n-i} (X(k) - \bar{X})(X(k+i) - \bar{X})}{\sum_{k=1}^{n} (X(k) - \bar{X})^2}$$

Two issues in using the above estimator:

(2) When i is close to n the number of product in the numerator is small and the estimate $\hat{\rho}(i)$ will not be good.

(3) Since $\hat{\rho}(i) \to 0$ when $i \to \infty$ we need to consider if any $\hat{\rho}(i)$ is statistically significant from zero.

Zhang (2006) proposed a practical approach to resolve these.

For (1), a rule of thumb: $\hat{\rho}(i)$ can only be used when $i \leq n/4$ and $n \geq 50$ Thus, we only use these $\hat{\rho}(i)$ with $i \leq n/4$

For (2), a cut off lag for $\hat{\rho}(i)$ is estimated based on the data.

An autocorrelated process or a time series can be non-stationary such as a ransom walk:

$$X(t) = X(t-1) + a(t)$$

with $X(0) = 0$ and $\{a(t)\}$ white noise. It is obvious that

$$X(t) = \sum_{i=1}^{t} a(i)$$

Thus,

$$Var[X(t)] = t\sigma_a^2$$

If measurements are from a non-stationary process using the average value and the corresponding variance to characterize the measurement standard may be misleading. For random walk, process variance increases with the time index.

As we have seen that for a stationary process such as AR(1), the variance of $\bar{X}$ decreases with a rate of 1/n when the sample size n increases. This property is used in metrology to reduce the uncertainty of repeated measurements

However, for a non-stationary process, the variance of $\bar{X}$ may not decrease or even may not always decrease when n increases. For random walk, the variance of $\bar{X}$ increases to infinity with a rate of n. Therefore, increasing the sample size will not be helpful to reduce uncertainty.

Based on this consideration, for some other processes in the area of time and frequency metrology Allan variance has been used as a substitute for the classical variance to characterize the stability of clocks or frequency standards as in Allan (1987) with a title of ”Should the classical variance be used as a basic measure in standards metrology?”.

In GUM, 4.2.7., it states that specialized methods such as the Allan variance were used to treat autocorrelated measurements of frequency standards.

It has been found that in time and frequency the random fluctuations in standards can be modeled by a power law of the power spectral density

$$f(\omega) = \sum_{\alpha=-2}^{2} h_\alpha \omega^\alpha$$

When a process has the property that $f(\omega) : 1/\omega$ when $\omega \to 0$ it is called 1/f noise. In general, a process is called a long-memory process if $f(\omega) \to \infty$ when $\omega \to 0$.

In Witt (2000, etc.) Allan variance was used to characterize the noise of Zener-diode voltage standards related to 1/f. But the process were restricted to the noises expressed in power law in time and frequency metrology. We will discuss the properties of Allan variance for a wide ranges of time series.

3. Allan variance for stationary processes

For $\{X(t)\}$, we define the process formed by arithmetic mean of n consecutive $X(t)$ or moving averages as

$$Y_n(1) = \frac{X(1) + \ldots + X(n)}{n} = \bar{X}_1$$

$$\ldots\ldots\ldots\ldots$$

$$Y_n(T) = \frac{X((T-1)n+1) + \ldots + X(Tn)}{n} = \bar{X}_T$$

The two-sample variance or Allan variance of $\{X(t)\}$ is defined as

$$AVar_n[X(t)] = \frac{E\{[Y_n(T) - Y_n(T-1)]^2\}}{2}$$

For a stationary process, the Allan variance can be expressed as

$$AVar_n[X(t)] = \frac{n[1-\rho(n)] + \sum_{i=1}^{n-1} i[2\rho(n-i) - \rho(i) - \rho(2n-i)]}{n^2}\sigma_X^2.$$

For a given a data set of $\{X(1),...,X(N)\}$, we can estimate its Allan variance.

The Allan variance is defined as

$$AVar_n[X(t)] = \frac{Var[Y_n(T) - Y_n(T-1)]}{2}$$

The Allan variance for the average size of n is estimated by

$$\dot{A}Var_n[X(t)] = \frac{\sum_{i=2}^{m} [Y_n(i) - Y_n(i-1)]^2}{2(m-1)}$$

for $i = 2,...,[N/n]$ and $m = [N/n]$. This is an unbiased estimator.

Allan variance for an i.i.d. sequence

$$AVar_n[X(t)] = Var[Y_n(T)] = \frac{\sigma_X^2}{n}$$

Allan variance for AR(1) processes

$$AVar_n[X(t)] = \frac{n - 3\phi_1 - n\phi_1^2 + 4\phi_1^{n+1} - \phi_1^{2n+1}}{n^2(1-\phi_1)^2}\sigma_X^2$$

$$= \frac{n - 3\phi_1 - n\phi_1^2 + 4\phi_1^{n+1} - \phi_1^{2n+1}}{n^2(1-\phi_1)^2(1-\phi_1^2)}\sigma_a^2 : \frac{1}{n}$$

Allan variance for a white phase process $X(t) - \mu = a(t) - a(t-1)$

$$AVar_n[X(t)] = \frac{3\sigma_a^2}{n^2} : \frac{1}{n^2}$$

Summary: For stationary and invertible AR processes, the variance of the sample averages and the Allan variance have the same convergent rate of 1/n when n approaches infinity. Therefore, they are similar measures for uncertainty of measurements from a stationary AR process.

4. Allan variance for other processes

4.1 Allan variance for random walk

For random walk, which is non-stationary

$$Var[Y_n(T)] = \frac{\sigma_a^2}{n} \frac{2n^2(3T - 2) + 3n + 1}{6} : n$$

$$AVar_n[X(t)] = \frac{2n^2 + 1}{6n} \sigma_a^2 : n$$

4.2 Allan variance for fractional difference ARFIMA(0,d,0)

$$(1 - B)^d X(t) = a(t)$$

where B is a back shift operator. When d is not an integer, it is called a fractional difference process.

Here $(1-B)^d$ is defined as

$$(1-B)^d = 1 - dB - \frac{d(1-d)}{2}B^2 - \frac{d(1-d)(2-d)}{6}B^3 - \ldots$$

When d < 0.5, it is stationary and

$$f(\omega) = \frac{\sigma_a^2}{2\pi\,(2\sin\frac{\omega}{2})^{2d}} : \omega^{-2d}$$

Thus, it is a long-memory process.

$$Var[Y_n(T)] = \left[1 + \frac{2\sum_{i=1}^{n-1}(n-i)\rho(i)}{n} \right] \frac{\Gamma(1-2d)}{[\Gamma(1-d)]^2}\frac{\sigma_a^2}{n} : n^{2d-1}$$

$$AVar_n[X(t)] = \frac{n[1-\rho(n)] + \sum_{i=1}^{n-1} i[2\rho(n-i) - \rho(i) - \rho(2n-i)]}{n^2} \frac{\Gamma(1-2d)}{[\Gamma(1-d)]^2}\sigma_a^2 : n^{2d-1}$$

When d = 0.5, it is not stationary. It is a flicker frequency process or a discrete 1/f noise.

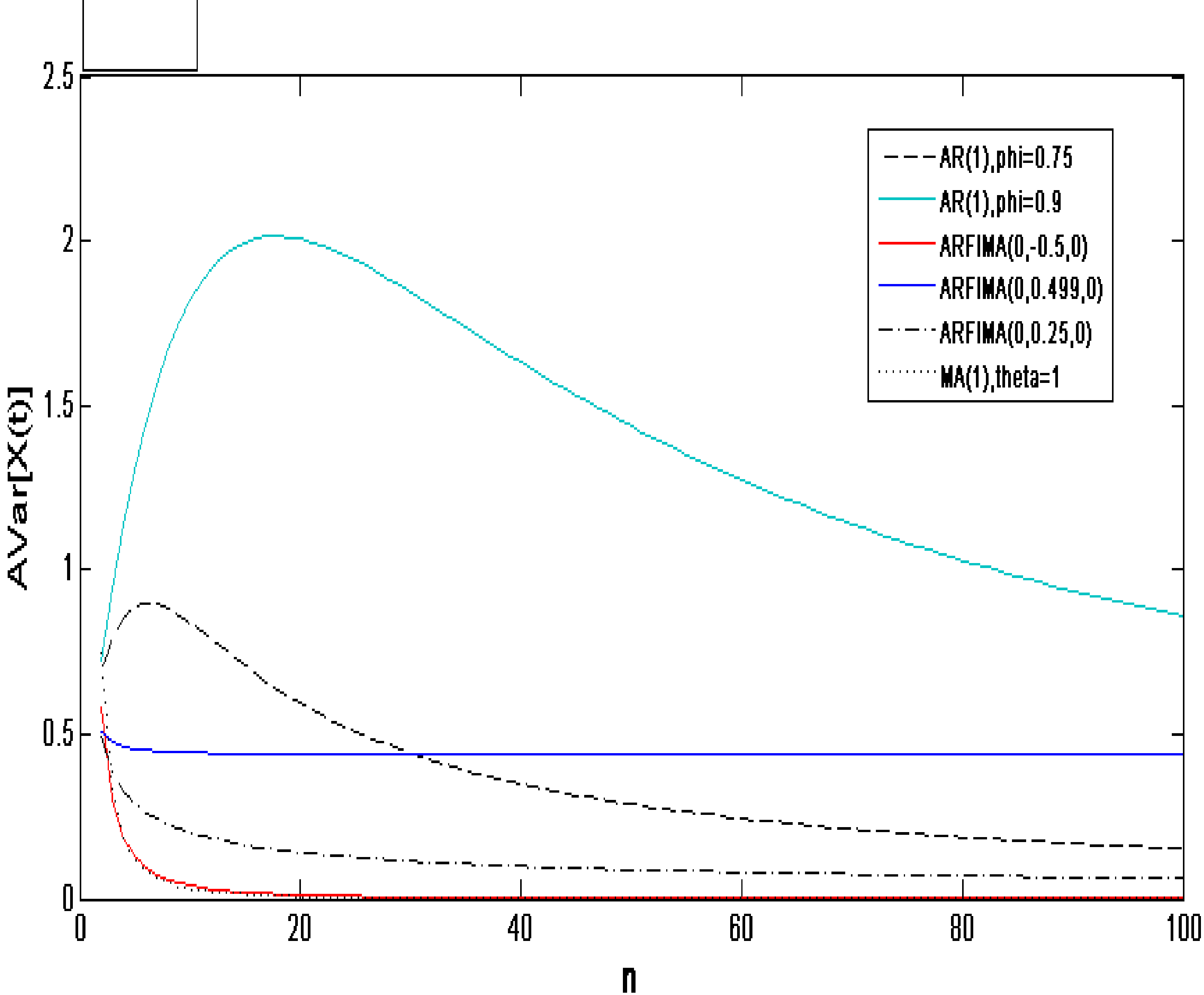
When $d \to 0.5$ $\quad Var[Y_n(T)] \to \infty$

However, when $d \to 0.5$ the Allan variance is finite and when $n \geq 100$

and $d \to 0.5$ the Allan variance approaches to $(2 \ln 2/\pi)\sigma_a^2$

To a great extent, it is consistent with the Allan variance of a 1/f in the frequency domain.

The property of a stabilized Allan variance or 1/f noise floor was exploited to test whether the measurements of the Zener-diode standards is 1/f noise.



AR(1),phi=0.75
AR(1),phi=0.9
ARFIMA(0,-0.5,0)
ARFIMA(0,0.499,0)
ARFIMA(0,0.25,0)
MA(1),theta=1
AVar[X(t)]
n
0
0.5
1
1.5
2
2.5
0
20
40
60
80
100

4.3 Allan variance for ARFIMA (p,d,q) processes

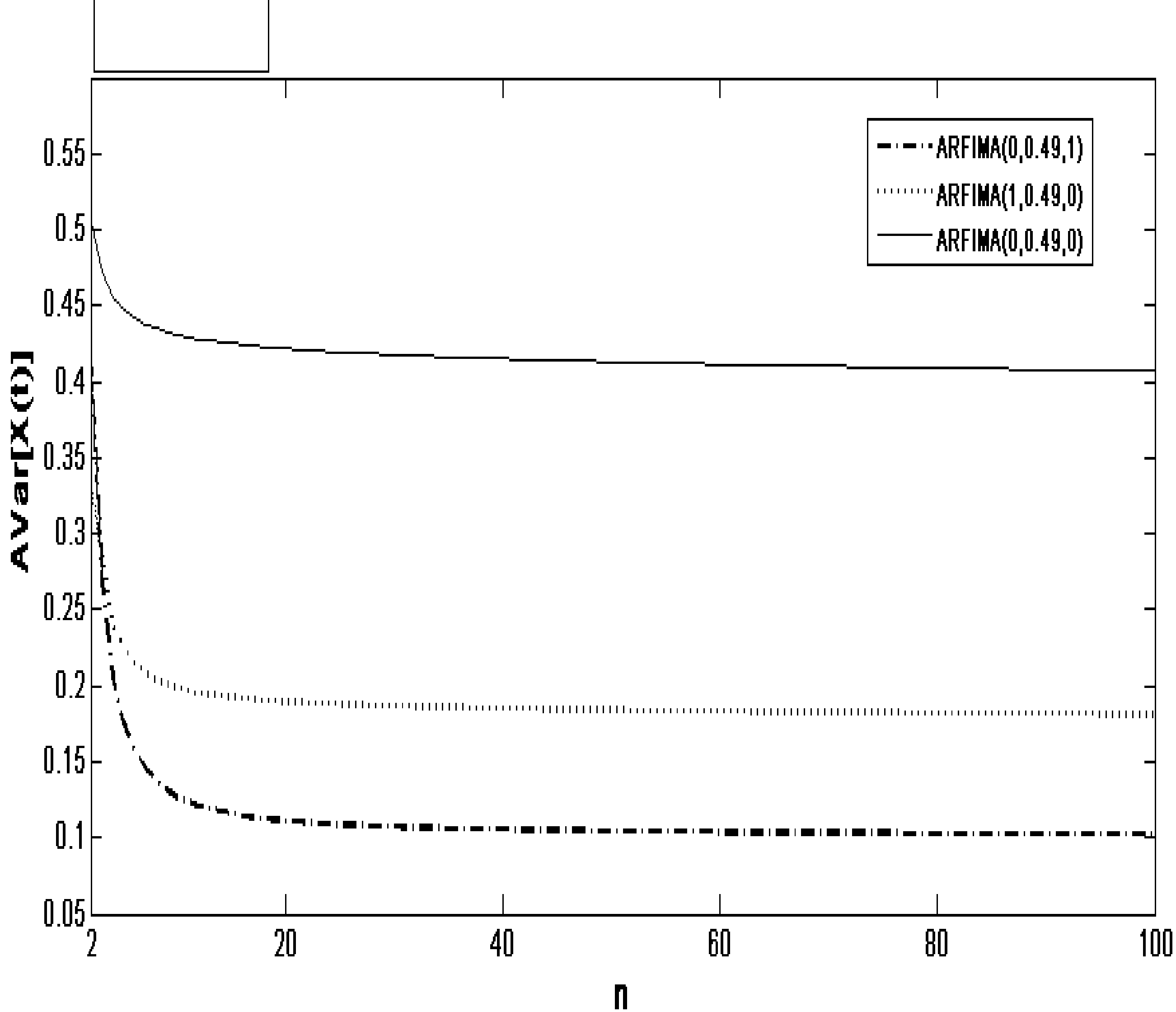
$$\phi(B)(1-B)^d X(t) = \theta(B)a(t)$$

Similar to ARFIMA(0,d,0), when d < 0.5, and some regular conditions for $\phi(B)$ and $\theta(B)$ are satisfied $\{X(t)\}$ is stationary.

In particular, for ARFIMA (1,d,0) and ARFIMA(0,d,1) the Allan variances can be calculated. For these processes, when $d \to 0.5$

$$Var[Y_n(T)] \to \infty$$

while the Allan variances are finite.



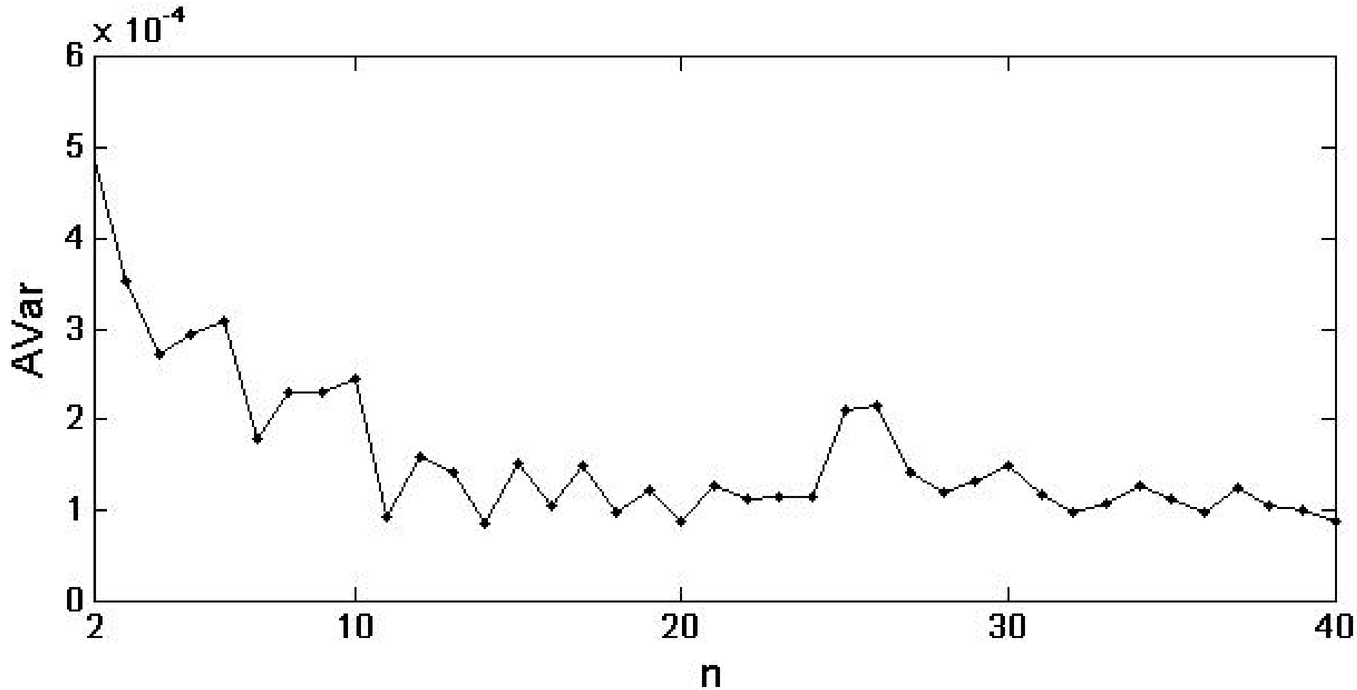
ϕ_1=-0.5 and $\theta_1 = 0.5$.

5. Examples

Example 1 – Weights measurement

The Allan variances are estimated $\widehat{Var}[Y_{217}(1)] = 0.0067^2$

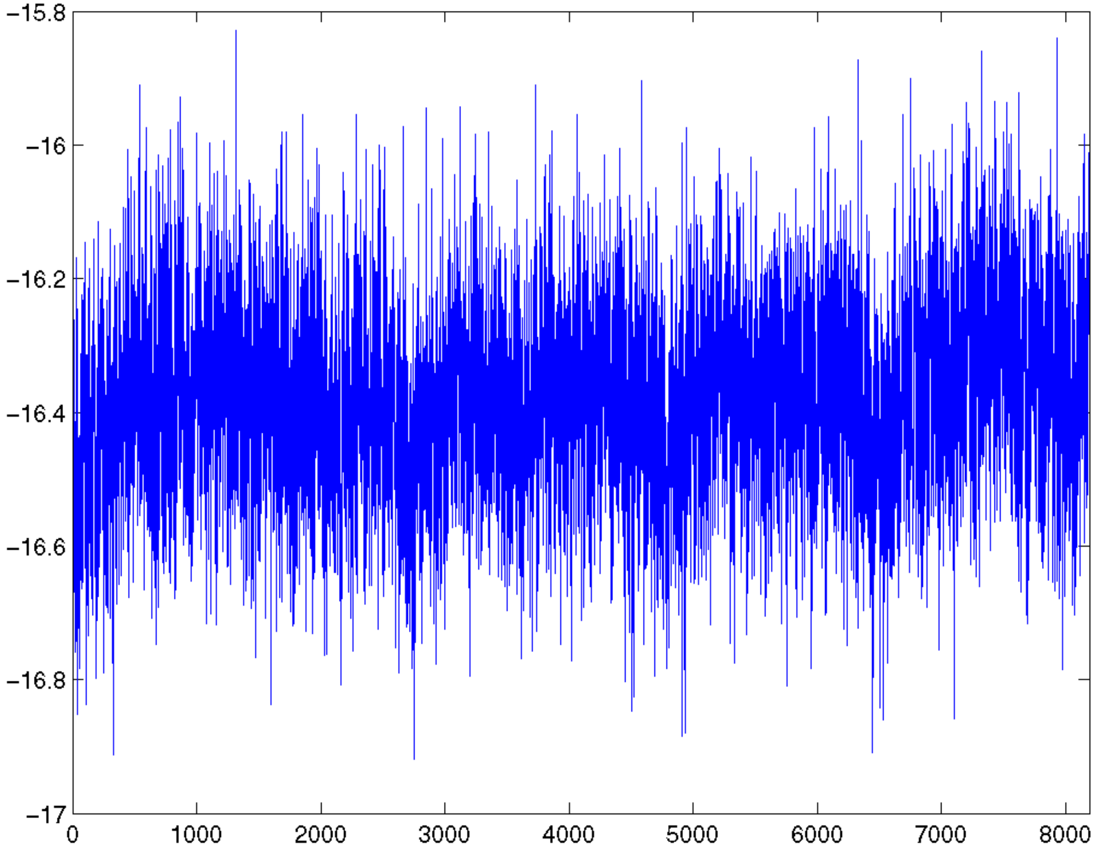
It is plotted against the average size.

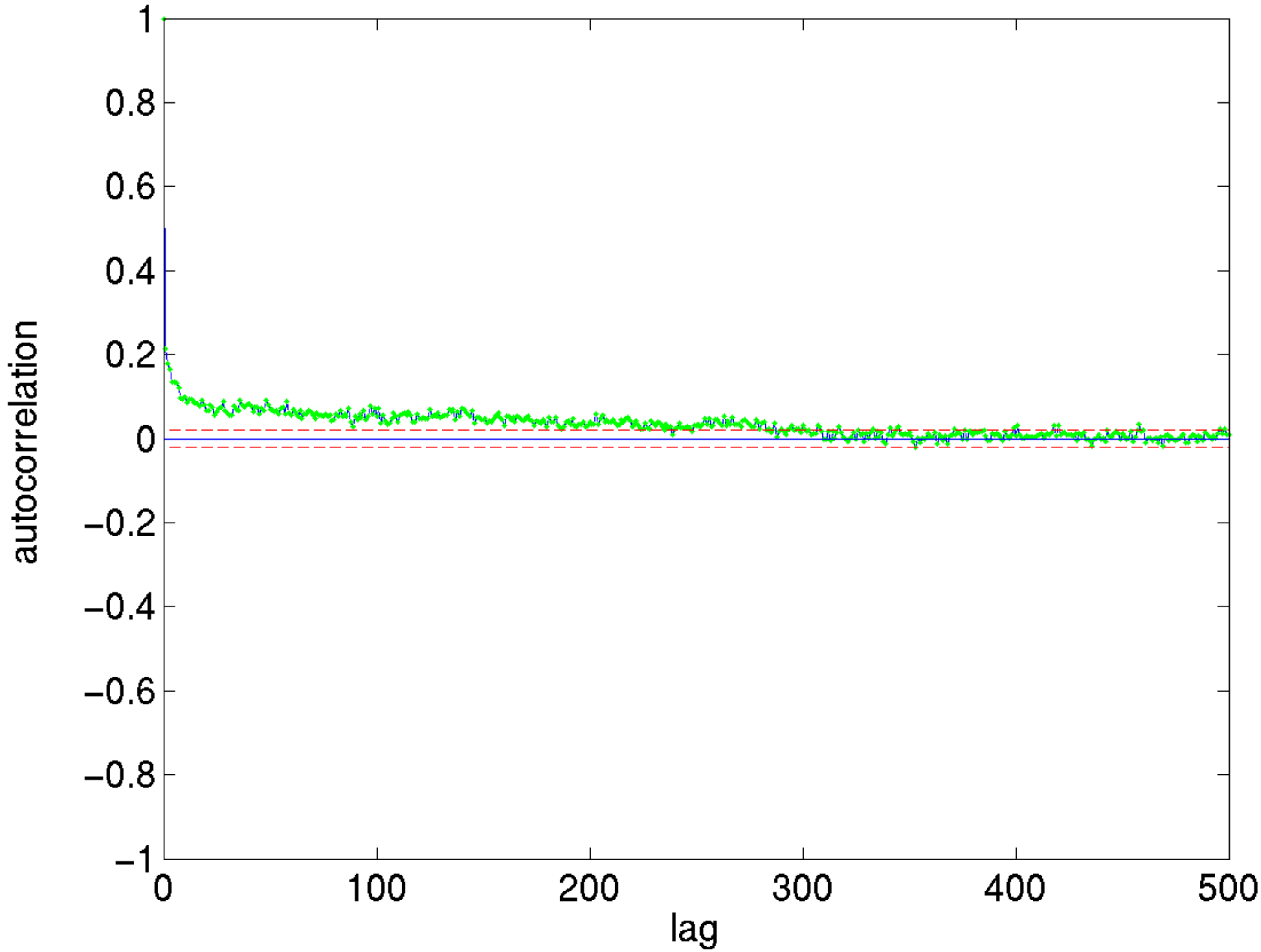


Obviously, the Allan variance decreases when increases.

Example 2

We show the behavior of a time series of a Zener voltage standard measured against a Josephson voltage standard. It has the differences of 8192 voltage measurements in the units of microvolt.



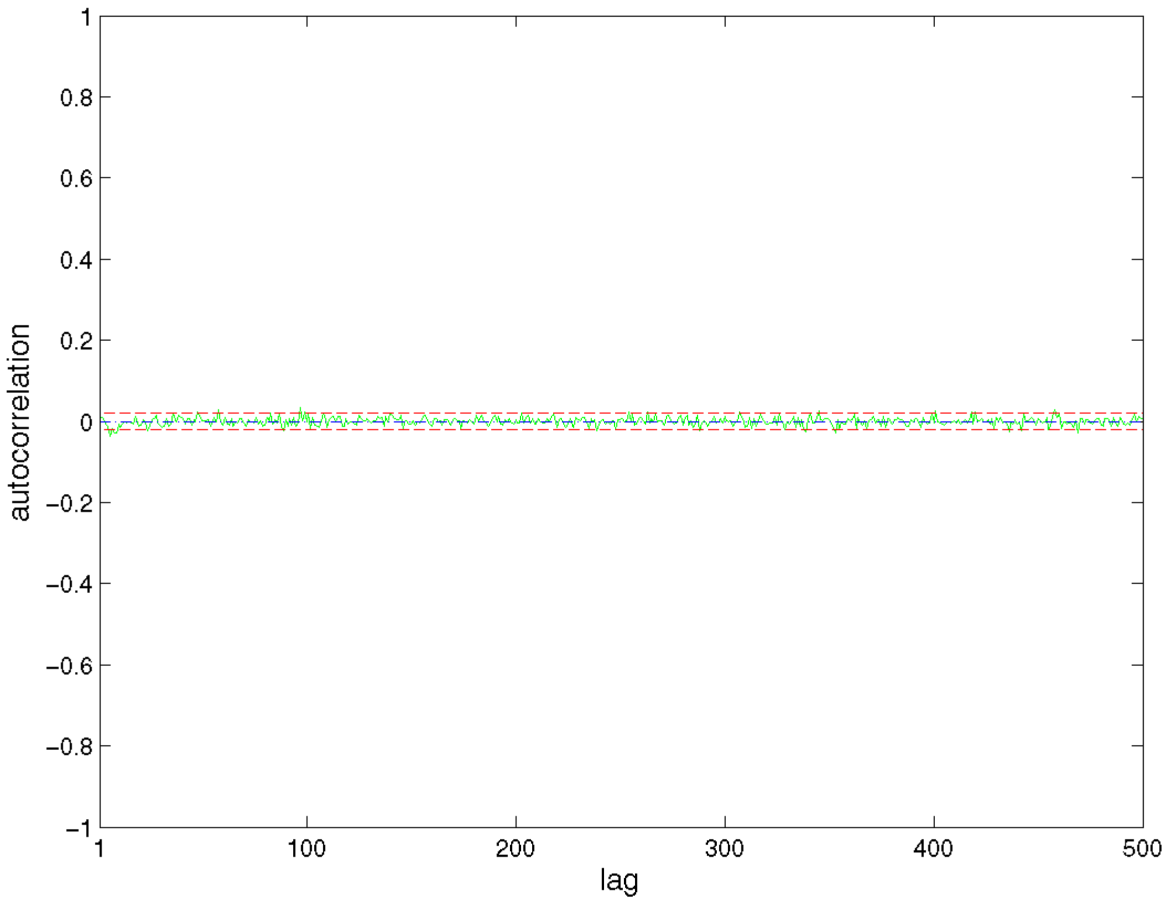


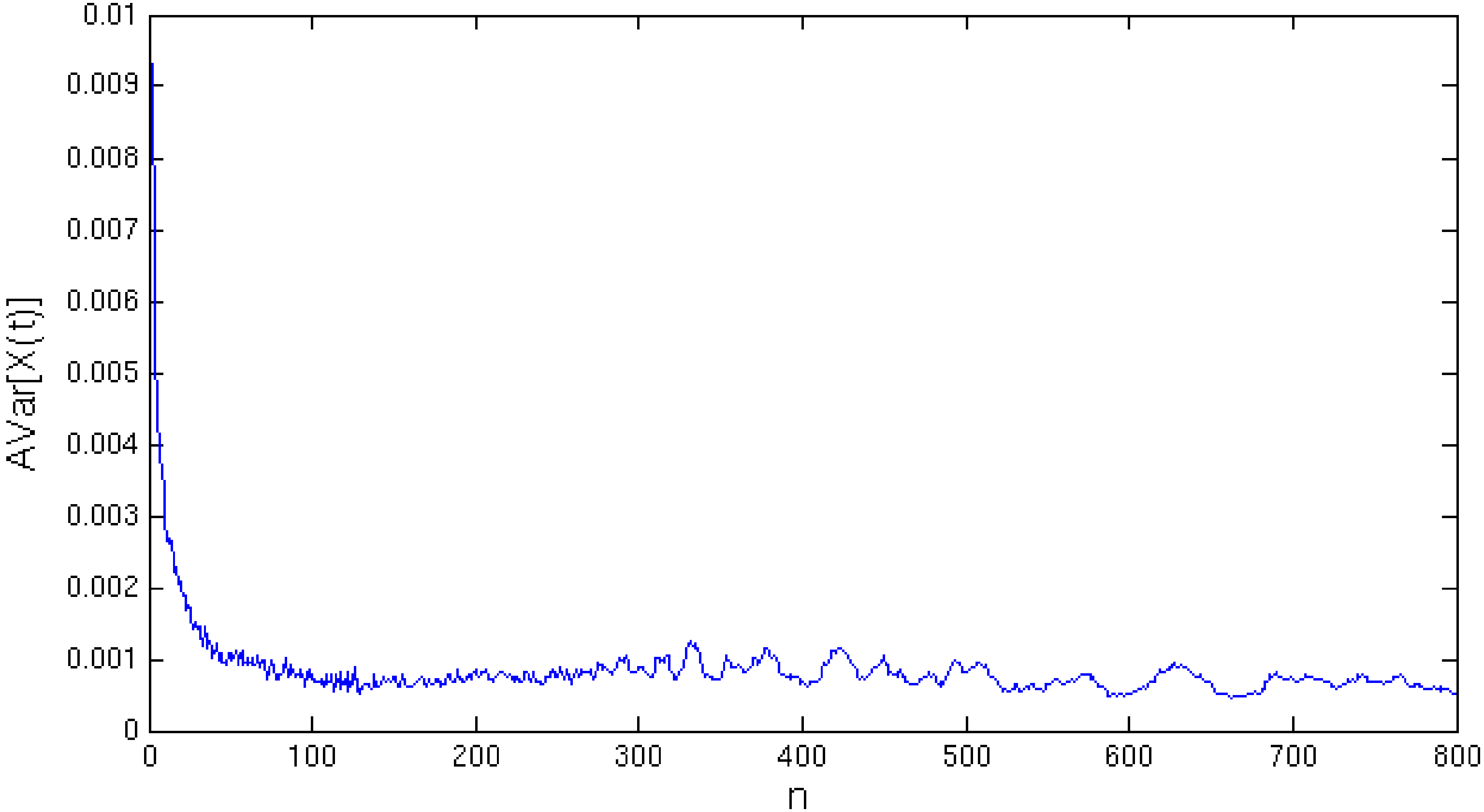
Sample autocorrelation function of the time series of the voltage differences

A model of ARFIMA(5,0.497,0) is a good fit with $\hat{\sigma}_a^2 = 0.02$

$$(1-B)^{0.497}[X(t)+0.35X(t-1)+0.2X(t-2)+0.1X(t-3)$$
$$+0.1X(t-4)+0.06X(t-5)+29.98]=a(t)$$

The sample autocorrelation of the residuals





Allan variance for the time series of voltage differences

6. Conclusions

- The variance of moving averages and the Allan variance of a stationary ARMA process, and a stationary fractional difference ARMA process are closely related. They decrease with a same rate when the size of the average increases.

- For a random walk, the variance of the moving averages and the Allan variance will go to infinity with a same rate when the size of the averages increases. However, the Allan variance of random walk is independent of the time index.

- For a non-stationary fractional difference process when $d = 0.5$ or $1/f$ noise the Allan variance is stabilized at a certain level while the variance of the moving averages will approach infinity. In this case, the Allan variance is a better uncertainty measure than the variance of the moving averages.

References

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