

Declaration and Specification of Geometrical Part in the language of Geometric Algebra

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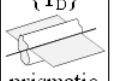
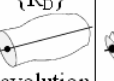
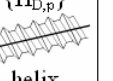
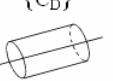

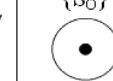
Motivation

- Success of “**Acausal**” models versus “**Causal**” models
 - acausal modeling is a convenient way to express **specifications**.
 - are easy to build and modify,
 - require highly elaborated tools to handle them efficiently,
 - causal modeling is a convenient way to express **explicit computations**.
 - are difficult to build and extremely hard to modify,
 - generally don't require elaborated tools to handle them efficiently,
- Today, users define complex objects. They must specify and solve multi-physics problems. Also problems with **physic and geometric** constraints.

Outlines

- Keywords in TTRS theory
- The Geometric Algebra (GA)
- The representation of TTRS in the GA language

- Deve
- Base
- **disp**
- It is
- subg

	{E}	{T _D }	{R _D }	{H _{D,P} }	{C _D }	{G _P }	{S _O }
	 any surface	 prismatic surface	 revolution surface	 helix surface	 surface cylindrical	 planar surface	 spherical surface
{E}	{E}	{E}	{E}	{E}	{E}	{E}	{E}
{T _D }		D ₁ // D ₂ Σ {T _{D1} }, otherwise Σ {E}.	{E}	{E}	D ₁ // D ₂ Σ {T _{D1} }, otherwise Σ {E}.	D ₂ // P ₁ Σ {T _{D2} }, otherwise Σ {E}.	{E}
{R _D }			D ₁ = D ₂ Σ {R _{D1} }, otherwise Σ {E}.	{E}	D ₁ = D ₂ Σ {R _{D1} }, otherwise Σ {E}.	D ₂ ⊥ P ₁ Σ {R _{D2} }, otherwise Σ {E}.	O ₁ ∈ D ₂ Σ {R _{D2} }, otherwise Σ {E}.

■ **solid**
in space.
in **twelve**

The number of TTRS is finite

CATIA CAD-system uses TTRS to represent Geometric tolerances

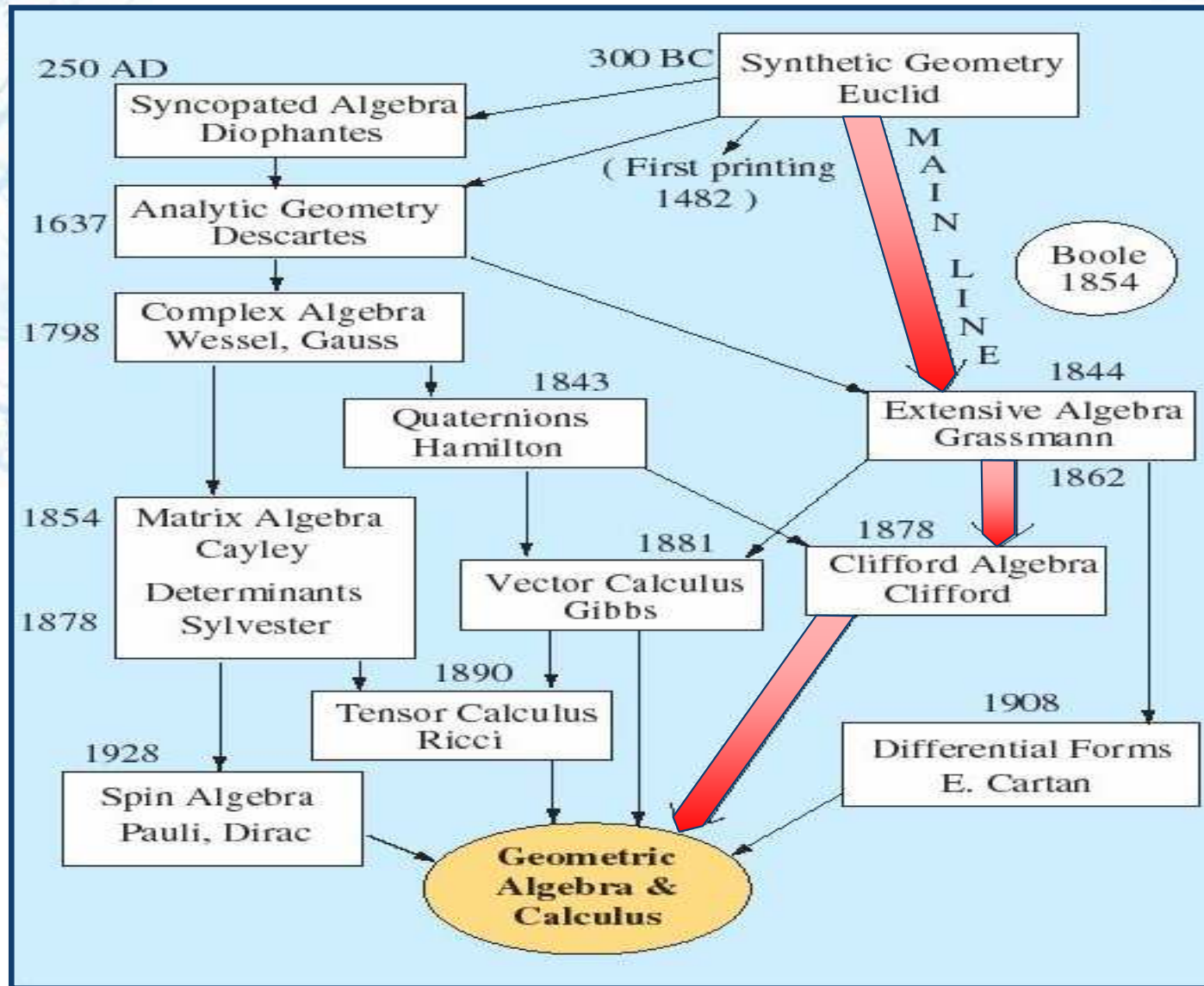
- Betw
- **13 r**

					otherwise Σ {E}.	Σ {E}.	
{G _P }						C6 : P ₁ // P ₂ Σ {G _{P1} }, C7 : otherwise Σ {T _D }.	C3 : {R _D }
{S _O }							C1 : O ₁ = O ₂ Σ {S _{O1} }, C2 : otherwise Σ {R _D }.

re only

The Origin of Geometric Algebra

The foundations of Geometric Algebra (GA) were established during the 19th century



What is the Geometric Algebra

1. It has an associative, noncommutative product with identity. The product is called the *geometric product*.
2. It is a vector space, called G^n , extending the vector space R^n .
 - It has 2^n dimension.
 - Its vectors are called *multivectors*.
 - Scalars are multivectors.
3. The geometric product (left and right) distributes over multivector addition.



Based on a single rule. For every vector \mathbf{u}

$$\mathbf{u}\mathbf{u} = \mathbf{u} \cdot \mathbf{u}$$

This equation shows that nonzero vector have an inverse

$$\mathbf{u}^{-1} = \mathbf{u} / (\mathbf{u} \cdot \mathbf{u})$$

For every vectors \mathbf{u} and \mathbf{v}

$$\left. \begin{aligned} (\mathbf{u} + \mathbf{v})(\mathbf{u} + \mathbf{v}) &= \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ (\mathbf{u} + \mathbf{v})(\mathbf{u} + \mathbf{v}) &= \mathbf{u}\mathbf{u} + \mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u} + \mathbf{v}\mathbf{v} \end{aligned} \right\} \rightarrow \mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u})$$

When \mathbf{u} and \mathbf{v} are orthogonal

$$\mathbf{v}\mathbf{u} = -\mathbf{u}\mathbf{v}$$

$$(\mathbf{u}\mathbf{v})^2 = \mathbf{u}\mathbf{v}\mathbf{u}\mathbf{v} = -\mathbf{u}\mathbf{u}\mathbf{v}\mathbf{v} = -(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) < 0$$

$\mathbf{u}\mathbf{v}$ is not a scalar
is not a vector,
it is a **bivector**

The vector space G^n has 2^n dimensions

■ 2D basis $\rightarrow 2^2=4$ elements


■ 1 scalar, 2 vectors and 1 bivector (a 2-vector)

$$\{1, e_1, e_2, e_1e_2\}$$

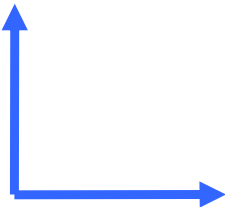
■ 3D basis $\rightarrow 2^3=8$ elements

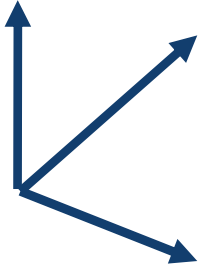
■ 1 scalar, 3 vectors, 3 bivectors and 1 trivector (a 3-vector)

$$\{1, e_1, e_2, e_3, e_1e_2, e_2e_3, e_3e_1, e_1e_2e_3\}$$


Grade 0
scalar


Grade 1
vector


Grade 2
bivector


Grade 3
trivector

Geometric product of two vectors

We investigate the geometric product of two given vectors \mathbf{u} and \mathbf{v} in \mathbf{R}^2 . Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be an orthonormal basis for a plane containing the vectors. Let $\mathbf{u} = a\mathbf{e}_1 + b\mathbf{e}_2$ and $\mathbf{v} = c\mathbf{e}_1 + d\mathbf{e}_2$. Then from the product rules,

$$\mathbf{u}\mathbf{v} = (ac + bd) + (ad - bc)\mathbf{e}_1\mathbf{e}_2$$

Inner Product

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$$

Outer Product

$$\mathbf{u} \wedge \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \sin \theta \mathbf{i}$$

$$\mathbf{u}\mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}$$

A *blade* \mathbf{B} for a subspace of \mathbf{R}^n is a product of members of an orthogonal basis for the subspace: $\mathbf{B} = \mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_k$. We call \mathbf{B} a *k-blade*, or a blade of *grade* k . Nonzero scalars are 0-blades.

Geometric Algebra represents subspaces with their blades

The geometric product between vector and k-blade is

$$\mathbf{u}\mathbf{B} = \mathbf{u} \cdot \mathbf{B} + \mathbf{u} \wedge \mathbf{B}$$

A k-blade as an inverse

$$\mathbf{B}^{-1} = \mathbf{b}_k \cdots \mathbf{b}_2 \mathbf{b}_1 / |\mathbf{b}_k|^2 \cdots |\mathbf{b}_2|^2 |\mathbf{b}_1|^2$$

TTRS theory

GA language

Is it possible to represent a TTRS with a multivector ?

we will give an answer in the following slides

But Why ?

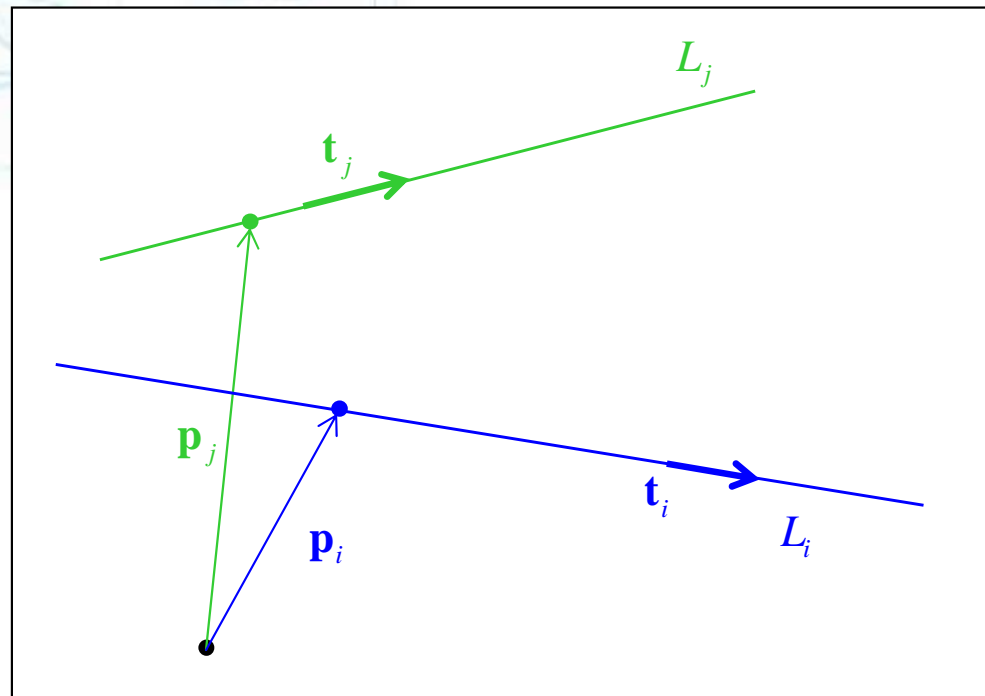
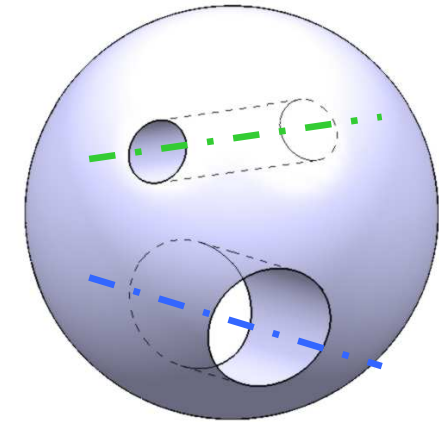
To solve the problem of the **chirality**.

To be more adapted with **declarative modeling**
in geometry domain.

To be extended toward the **multi-physics** problems.

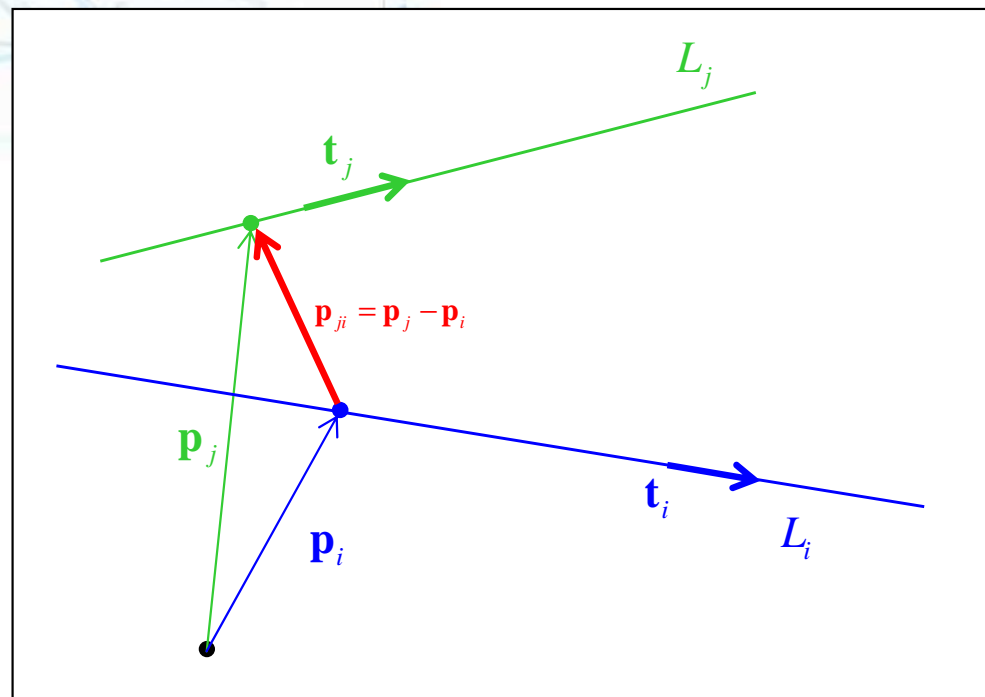
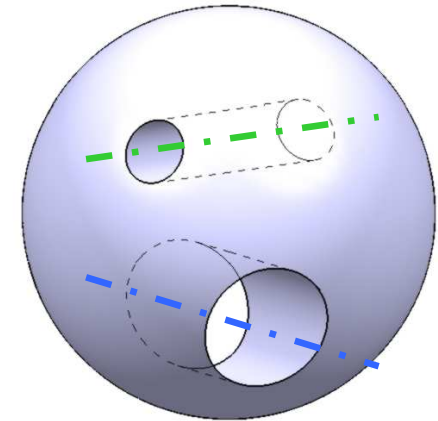
Each Line is represented by

- One point (Affine Space)
- One direction (Vector Space)
- The multivector $\mathbf{t}_i + \mathbf{p}_i \mathbf{t}_i$



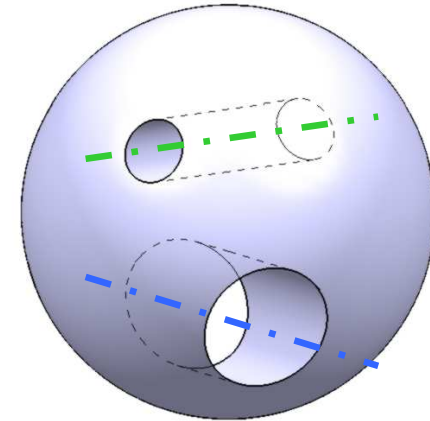
Line-Line Association

- Affine Space \rightarrow Set of points
- Vector Space
- Association of two directions, and
- **New vector for the affine connection**



Line-Line Association as member of GA

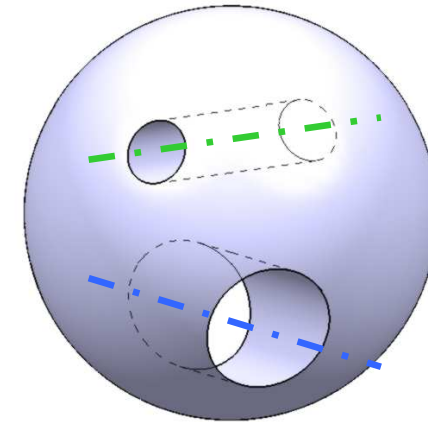
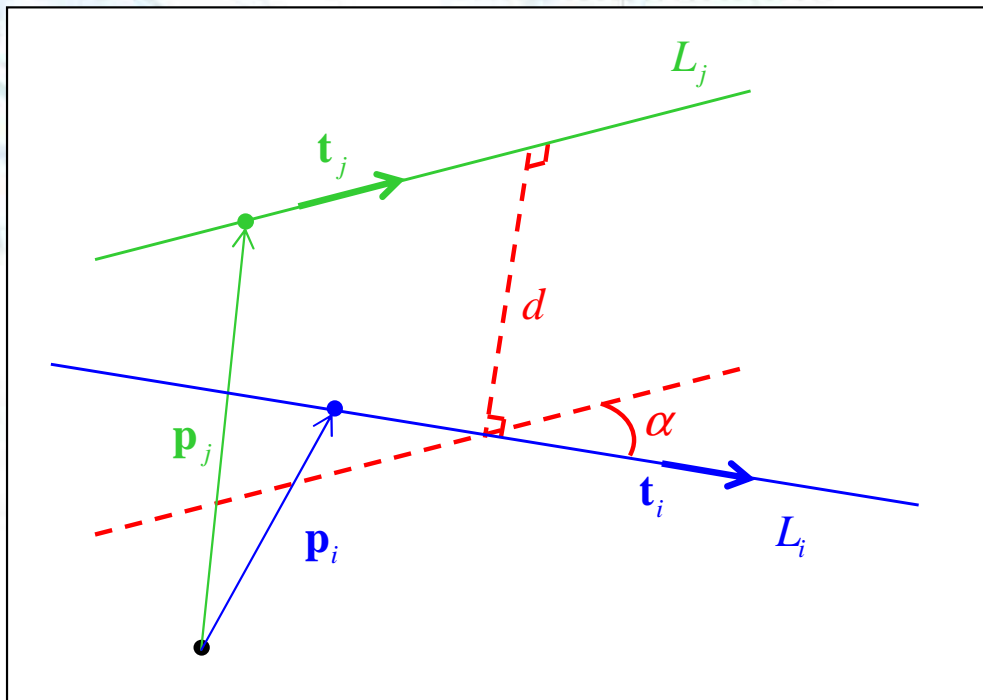
- Affine Space → Two points
- Vector Space → Three directions
- The following multivector



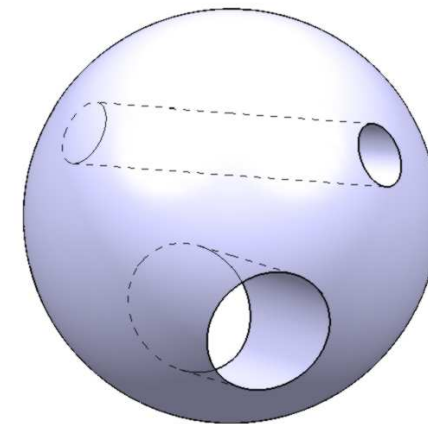
$$\underbrace{\mathbf{t}_i \mathbf{t}_j}_{\text{Tangent}} + \underbrace{\mathbf{p}_{ji} \mathbf{t}_i \mathbf{t}_j}_{\text{Moment}}$$

Grade: 0 ; 2 Grade: 1 ; 3

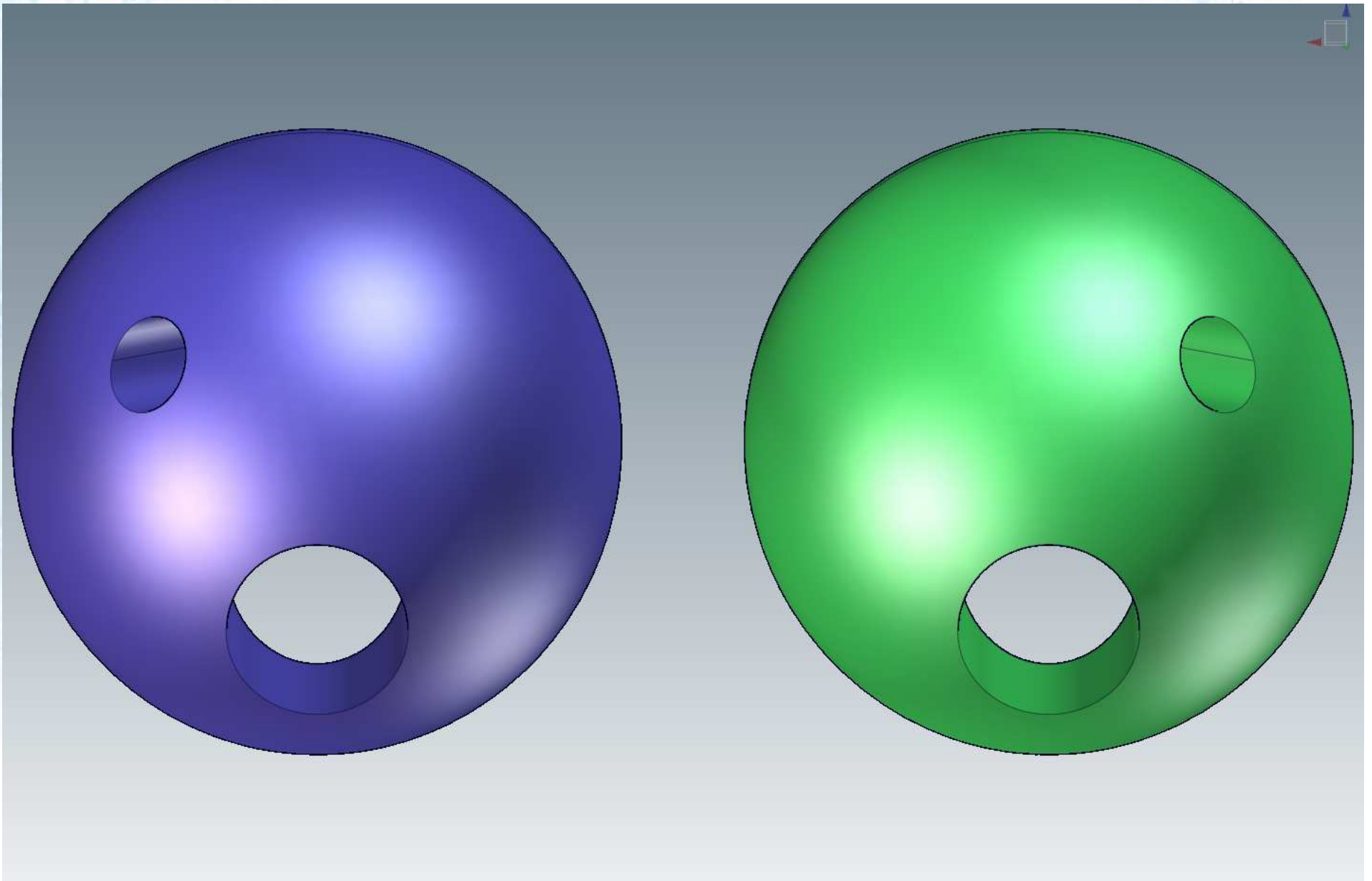
Usually, to specify this object, a designer provides the distance (along the common perpendicular) and the angle between the two lines.



« Left » Hand

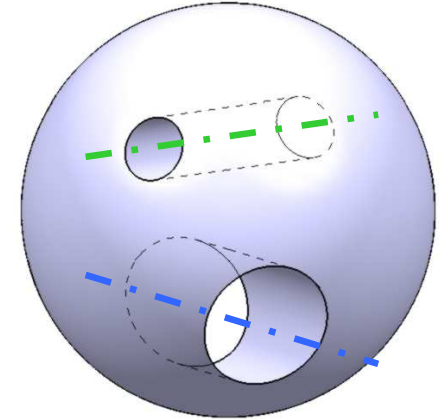


« Right » Hand



TTRS is represented by a multivector

$$\mathbf{t}_i \mathbf{t}_j + \mathbf{p}_{ji} \mathbf{t}_i \mathbf{t}_j$$



$$\frac{(\mathbf{t}_i \wedge \mathbf{t}_j)}{(\mathbf{t}_i \cdot \mathbf{t}_j)} = \tan \alpha \mathbf{I}_2$$

Angle

$$\frac{(\mathbf{p}_{ji} \wedge \mathbf{t}_i \wedge \mathbf{t}_j)}{(\mathbf{t}_i \wedge \mathbf{t}_j)} = du$$


Distance

$$(\mathbf{p}_{ji} \wedge \mathbf{t}_i \wedge \mathbf{t}_j)$$

Chirality

Association Table between Point, Line and Plane

Based on the same simple principle

	Point $\{\mathbf{p}_j; 0\}$	Line $\{\mathbf{p}_j; \mathbf{t}_j\}$	Plane $\{\mathbf{p}_j; \mathbf{B}_j\}$
Point $\{\mathbf{p}_i; 0\}$	$D_1 = \mathbf{p}_{ji}$	$D_2 = \mathbf{p}_{ji} \mathbf{t}_j$	$D_3 = \mathbf{p}_{ji} \mathbf{B}_j$
Line $\{\mathbf{p}_i; \mathbf{t}_i\}$	$D_4 = \mathbf{p}_{ji} \mathbf{t}_i$	$D_5 = \mathbf{t}_i \mathbf{t}_j + \mathbf{p}_{ji} \mathbf{t}_i \mathbf{t}_j$	$D_6 = \mathbf{t}_i \mathbf{B}_j + \mathbf{p}_{ji} \mathbf{t}_i \mathbf{B}_j$
Plane $\{\mathbf{p}_i; \mathbf{B}_i\}$	$D_7 = \mathbf{p}_{ji} \mathbf{B}_i$	$D_5 = \mathbf{B}_i \mathbf{t}_j + \mathbf{p}_{ji} \mathbf{B}_i \mathbf{t}_j$	$D_5 = \mathbf{B}_i \mathbf{B}_j + \mathbf{p}_{ji} \mathbf{B}_i \mathbf{B}_j$

 Case study

We have demonstrate

- Geometric Algebra is efficient to specify Geometric Objects.
- TTRS theory assures that the number of cases studies is finite.
- The TTRS between Plan, Line or Plane can be represented by a multivector. Mains advantages are
 - One TTRS \Leftrightarrow One multivector
 - All the geometric properties are included
 - The chirality is a specification (not a result)

Further work

- And the others TTRS ...
- The approach must be extended to represent and solve "Physic and Geometric" problem.

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